Mathematical Reviews

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Mathematical Reviews

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HISTORY AND BIOGRAPHY

See also 6716, 7072.

6311:

★**Мандрыка, А. П. Баллистические исследования Леонарда Ейлера.** [Mandryka, A. P. **Ballistic investigations of Leonard Euler.**] Under the editorship of B. N. Okunev. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1958. 185 pp. 9.60 rubles.

This is one of the volumes issued in connection with the 250th anniversary of Euler's birth. Euler is a mathematician whose work is scarcely known at all, not because it is unpublished, difficult to obtain, or even difficult to understand, but simply because it is unread. Everyone who tries to write seriously about some part of Euler's work finds that to do it justice, he must write the whole history of the subject in the period, for Euler nearly monopolized every subject he touched.

The present book, no exception, seems to be the first serious history of mathematical ballistics in the eighteenth century. While only about 2% of Euler's work concerned this subject, he left a permanent mark upon it, and several of his results are standard in ballistic usage today. It will be recalled that Euler translated into German the booklet on experimental ballistics by his detractor, Robins, and added to it notes, mainly theoretical, doubling its bulk and transforming it into the first scientific treatise in that field (1745). All of Euler's work on interior ballistics is in this treatise and hence is earlier than the discovery of the equations of fluid dynamics. Much of his work on exterior ballistics is even earlier than the treatise. Euler did not follow here his custom of writing later in life works going over his early studies, incorporating them in more general and more analytic presentations.

The present volume gives a detailed account of theoretical work done in ballistics from 1700 to 1760, roughly, and includes also some aspects of the theories of resistance experienced by submerged bodies, of the spring of air, and of the nature of fire. Abridged table of contents: General characteristics of Euler's work on ballistics; Chapter I (Exterior ballistics): method of John Bernoulli and Euler (1719–1736), flow around an obstacle in Euler's treatise (1745), Robins' experiments and Euler's binomial resistance formula, Euler's method of defining the elements of motion for a projectile fired horizontally, Euler's study of the motion of a projectile fired vertically upwards, Euler's solution of the fundamental problem of exterior ballistics using his binomial resistance formula (1745), Euler's method in the case of a quadratic law of resistance (1753); Chapter II (Interior ballistics): concept of elastic force of gunpowder in the work of Robins and Euler, Euler's theory of the elasticity of gases, Robins' method for determining muzzle velocity, Euler's methods for determining muzzle velocity on the assumption that the charge is burned up instantaneously or gradually, Euler's theory of the emanation of gunpowder gas, Euler's method for determining the optimum weight of charge and length of barrel. *C. Truesdell* (Bloomington, Ind.)

6312:

★**Бернулли, Даниил. Гидродинамика или записки о силах и движениях жидкостей.** [**Bernulli, Daniil=Bernoulli, Daniel. Hydrodynamics, or studies on the forces and motions of fluids.**] Title also in Latin. Translation by V. S. Gohman; commentary and redaction by A. N. Nekrasov and K. K. Baumgart; essay by V. I. Smirnov. Izdat. Akad. Nauk SSSR, 1959. 551 pp. 23.75 rubles.

Daniel Bernoulli's name is familiar to every mathematician, but few have any knowledge of his work beyond the vague generalities, inaccurate details, and dubious anecdotes handed down in popular histories. Most mathematical libraries in the U.S.A. do not contain a single work by him. His papers have never been republished in a collected edition; to the best of this reviewer's knowledge, Bernoulli's great classic work, the *Hydrodynamica*, is here printed only for the second time, the date of the first edition being 1738. The contents of the book are so completely unknown among mathematicians and specialists in fluid mechanics that a detailed review, as if it were a new work, would be appropriate, but there is not space here to do it justice. A few of the most important paragraphs have been translated into English and discussed by the reviewer [Part IV of Introduction to *L. Euleri opera omnia*, ser. 2, vol. XII, Soc. Sci. Nat. Helveticae, Lausanne, 1954; MR 17, 2]; it is to be hoped that the edition now underway in Basel will not be too long delayed.

The present volume contains a Russian translation of the text and 45 pages of annotations. Some of these explain the mathematical steps in the notoriously difficult text; others give relevant passages and sketches from previously unpublished documents, including the first draft Bernoulli left behind him in Petersburg in 1733. It is a pity the Latin originals are not given along with the Russian translations. Following the text is the brief autobiography of Bernoulli and a valuable and concrete essay, 70 pages long, on his life and works by V. I. Smirnov.

Daniel Bernoulli has stood in the shadows of his overbearing father and the dazzling Euler, both of them far quicker and more versatile than he. However, he had a strong and original mind, independent to the point of blindness to the work of even his nearest colleagues. Certain philosophers of science could point to him and to his idol, Huygens, with more justice than to Galileo and Newton as heroes in the control of theory by experiment. However, his work extended into pure mathematics, especially very early and very late in his life; for example,

in his last decade he made the capital discovery that a trigonometric series can represent a discontinuous function.

The eighteenth century is the least known period in the history of the mathematical sciences among those for which adequate source material exists. The present volume is a useful contribution to its small historical literature. *C. Truesdell* (Bloomington, Ind.)

6313:

★**Turnbull, H. W. (Editor) The correspondence of Isaac Newton. Vol. I, 1661–1675.** Published for the Royal Society by the Cambridge University Press, New York-Cambridge, 1959. xxxviii+468 pp. (7 plates) $25.00.

No great mathematician is so difficult to study as Newton. For Galileo, Kepler, Descartes, and Huygens, monumental editions have printed, explained, and interconnected every remaining scrap of paper; for the Bernoullis and Leibniz, such editions have begun; for Lagrange, Laplace, and Cauchy, all the published works have been reissued in collected editions, in which some letters and personal documents were included; and the great Euler edition of 72 volumes, also limited to republication, is nearing its end; while every nineteenth-century figure with any kind of a name (except, unfortunately, for Navier, Poisson, and St. Venant) has been honored by a collection. The only edition of the works of Newton was printed nearly 200 years ago; not only far from complete, it is not trustworthy in what it does print. There is not even a critical edition of the *Principia*, let alone a collection of documents and analyses revealing its growth and antecedents. The text of the only easily available English translation is revised without understanding and annotated with puerilities. In such lack of available source material, it is no wonder that the histories of mathematics continue to parrot hoary hagiography and Victorian twaddle about the man who is usually made the hero of "the scientific revolution".

There have been several projects of a great collected edition, but all have failed. The initiates always give reasons, and such reasons are adduced in the opening pages of the introduction to the present volume by E. N. da C. Andrade, but to non-initiates in the Newton problem they may seem organizational rather than real, in view of the difficulties, apparently no more "intractable", that have been so brilliantly overcome, indeed by great devotion and at some cost, in the editions of Huygens and Galileo. It is sometimes conjectured that the main block to an edition of Newton's papers in full is the fear that they would draw a portrait different from that which three centuries of Newtonian myths have inculcated.

Be that as it may, everyone should welcome the appearance of the first volume of an edition, more modest in scope but certainly completable in short order, of all of Newton's correspondence. According to the foreword, three volumes have been prepared for the press by a large committee of experts working for thirteen years under the direction of H. W. Turnbull; later announcements indicate that there are to be seven volumes in all. It is estimated that about 430 letters by Newton and well over 1000 letters addressed to Newton or touching upon his work will be printed.

Of the 156 items in the present volume, covering the years 1661–1675, only 19 have not been printed before, and of these 19, only four are by Newton. Nevertheless, the contents of the volume make, in effect, a major new fund of source material. First, some of the previous publications are old and difficult to locate, so that the letters printed therein will not have been seen before except by persons who have searched for them. Second, some of the older publications are inaccurate from carelessness or censorship. Third, and most important, this is the first time that anything like a consecutive and connected view of any part of Newton's activity becomes possible, and this material all in one place, properly explained by excellent notes, is of far greater use and value than the sum of its formerly scattered parts. The editors have faced the realities of modern learning by appending English translations or paraphrases of all passages written originally in a learned language.

The main correspondents are Collins, Gregory, and Oldenburg. Oldenburg, secretary of the Royal Society, transmitted information as he saw fit to various others, including Huygens, Leibniz, and Hooke. Readers will welcome particularly the editor's decision to print relevant letters or passages from contemporary correspondences related to Newton's work and to the background of it; these items, making up perhaps a third of the volume, double the value of the rest by making it comprehensible.

The origin of the hostility between Newton and Hooke, sometimes regarded as a principal cause of Newton's reticence to communicate and hence a principal factor in the decline of British mathematics in the century following his death, may be seen here: Nos. 40, 44, 45, 67, 71 (not previously printed), 152, 154. As has been remarked before, Oldenburg disregarded Newton's request (No. 68) to soften any provocative expressions in his replies before transmitting them, and it seems that Oldenburg kept the matter hot by design. But also Hooke's evaluation (No. 44) of Newton's theory of light was written by order of the Royal Society, and the account of the matter given by Birch implies that the Society, by attaching so great importance to itself and to its aegis, operated in such a way as to make personal warfare almost inevitable among any creative personalities within it.

The main scientific subjects are optics, which it would be inappropriate to discuss here, and three parts of pure mathematics: numerical solution of equations, infinite series, and quadrature. Most of the mathematics is contained in the correspondence between Newton and Gregory through the intermediacy of Collins. Turnbull writes, "... in contemporary judgment the genius of Gregory was reckoned to be second only to that of Newton, a judgment with which those who have studied the available facts would probably still agree." For example, in No. 24 Gregory communicates the beginnings of seven series for transcendental functions such as $r \log \tan (\pi/4 + e/2r)$, with the statement, "I have no inclination to publish any thing, safe only to reprint my quadrature of the circle, and to add some litle trifles to it." Turnbull says, "Gregory had now discovered the method of successive differentiation which produces the Taylor expansion ..." In reply (No. 26), Newton stated an elegant rule for the sum of a finite number of terms of $\sum a/(b+nc)$, with bounds of error. The most interesting previously unpublished mathematical piece, No. 90, is also from this correspondence. It contains Newton's

ingenious substitute for a calculation which later mathematicians would have carried out by a double integral; an equally ingenious method for solving algebraic equations by laying a rule across parallel logarithmic scales; and the construction of conics by movable rulers.

These are but a few of the interesting mathematical problems and methods occurring. The reader will remark the extraordinary intensity of the mathematical spirit at this time, one rarely encountered before or since.

The volume is handsomely printed but not bound stoutly enough for a work of permanence. The editing is all that could be desired to make the volume easily readable and immediately useful to a mathematician. The series will be a standard work, needed by any solid library in the mathematical sciences.

C. Truesdell (Bologna)

GENERAL

6314:

★Sloan, Robert W. An introduction to modern mathematics. Prentice-Hall Mathematics Series. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1960. xi+73 pp. $3.75.

A very brief description of the simplest ideas of sets, logic, and functions. *K. O. May* (Northfield, Minn.)

6315:

★Minrath, William R. Handbook of business mathematics. D. Van Nostrand Company, Inc., Princeton, N.J.-Toronto-London-New York, 1959. xii+658 pp. $9.85.

The author has filled a long-existing void in the field of business mathematics by combining basic mathematics and their application to practical business problems. The first part of the book, which deals with basic mathematics, covers topics ranging from fractions to differential and integral calculus. Each topic is presented in a clear and logical fashion that is well adapted for either classroom instruction or easy reference. The major section of the book presents the methods of solving business calculations and covers all common areas from simple interest and commercial discount to the mathematics of life annuities and life insurance. The chapters on annuities, depreciation, securities, installment loans and real estate not only contain mathematical procedures and reference tables but much valuable background information. The concluding chapters are devoted to a discussion of the role of machines in business computation, operations research and the theory of games.

The author's lucid presentation reflects his well-rounded and wide experience in dealing with mathematical presentations as well as with practical business and financial problems. *R. Wubbles* (New York, N.Y.)

6316:

★Минорский, В. П. Сборник задач по высшей математике. [Minorskiĭ, V. P. Collection of problems in higher mathematics.] 5th ed. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 359 pp. 6.10 rubles.

The book contains 2570 exercises, with answers, on the following topics: plane analytic geometry, vector algebra, solid analytic geometry, higher algebra, introduction to analysis, derivative and differential, applications of the derivative, indefinite integral, definite integral, curvature of plane and skew curves, partial derivatives and total differentials, differential equations, multiple and curvilinear integrals, series.

6317:

Meyers, Leroy F. An integer construction problem. Amer. Math. Monthly **66** (1959), 556–561.

The author proves two general theorems, from which he deduces the following particular result: The positive integers obtainable from the numbers 1, 2, 3, 4, each being used once, by the operations of addition, subtraction, multiplication and negation (i.e. putting a 'minus' sign before a number), are the integers from 1 to 28 and the numbers 30, 32 and 36. *T. Estermann* (London)

LOGIC AND FOUNDATIONS

See also 6603.

6318:

★Popper, Karl R. The logic of scientific discovery. Hutchinson and Co., Ltd., London, 1959. 480 pp. 50s.

This book is an English translation of *Logik der Forschung* [Springer, Vienna, 1935], plus about 160 pages of material published or written by Popper since 1934, and there are also several references to a forthcoming supplementary book. The resulting impression that the past, present, and future are discussed almost simultaneously presents the reviewer with a problem. I shall try to solve this problem by making some references to Popper's other publications. It will be more useful if I concentrate on points of disagreement than on points of agreement. Popper's arguments are often lucid and powerful and can speak for themselves, but it would be a pity if the less good arguments should stand unanswered.

Some of the main subjects discussed are probability, induction, and quantum mechanics. The discussion is interesting and provocative, and often highly polemical. Popper believes that the best thing to do with a theory is to try to refute it, and when he thinks he has a refutation he says so emphatically.

Popper states that an analysis of knowledge by way of an analysis of ordinary language is too narrow, and is bound to miss the most interesting problems, which are those of science. On the other hand he has no sympathy with the construction of miniature model languages for science. "These model languages have no bearing on either science or common sense." This remark, like many others in the book, is dogmatic; the work on model languages is still in its infancy: it is easier to sit on a baby than to baby-sit. On the other hand I sympathise with the declaration of faith: "... I am interested in science and in philosophy only because I want to learn something about the riddle of the world in which we live, and the riddle of man's knowledge of that world. And I believe that only a revival of interest in these riddles can save the sciences and philosophy from narrow specialization and

from an obscurantist faith in the expert's special skill and in his personal knowledge and authority." A scientist who is not a philosopher is a mere technician.

Popper comes from Vienna, but he is not a positivist and he produces very strong arguments against positivism. He is not against metaphysics, but he rightly considers it important to distinguish metaphysical statements from scientific ones. His criterion is that metaphysical statements are not falsifiable. [This may be compared with the reviewer's criterion that there is no conceivable experiment that can greatly change the logarithm of the odds; see I. J. Good, Science **129** (1959), 443–447.] In his "Philosophy of science: a personal report" [in *British philosophy in the mid-century*, ed. by C. A. Mace, 1957, esp. p. 157] he gives the example of Freud's theory: "Whatever happens always confirms it." What the Freudians must do in order to answer Popper's criticism is to convert Freud's theory into a statistical hypothesis, so that it makes probabilistic predictions capable of refutation in principle.

Popper considers that the principle of scientific induction is unnecessary, unhelpful and inconsistent (p. 53), and proposes to replace it by a criterion of falsifiability of hypotheses. He does not believe that universal statements have a probability, or, if they have, the probability is zero. He says (p. 258) "... the probability of a hypothesis would depend on the training and skill of the experimenter rather than upon objectively reproducible and testable results" and that this "would make the probability of a hypothesis hopelessly subjective". (The reviewer has no sympathy with the use here of the word "hopelessly".) It may be observed that Popper's views are conformable, in statistics, with the use of null hypotheses and tail-area probabilities, rather than with Bayesian or neo-Bayesian methods, although the expressions "null hypothesis" and "tail-area probability" are not in the Index.

On p. 369 Hume's argument against scientific induction is quoted. In Hume's terminology the principle is that "those instances of which we have had no experience, are likely to resemble those of which we have had experience". Hume said: "Nay, I will go farther, and assert that he could not so much as prove by any probable arguments that the future must be conformable to the past. All probable arguments are built on the supposition that there is conformity betwixt the future and the past, and therefore can never prove it." The reviewer's opinion is that the principle of scientific induction is a consistent metaphysical hypothesis, reasonably probable, and extremely useful, and it should be accepted because its expected utility is large and positive.

Popper produces a "more direct disproof" of the principle. He takes an argument of H. Jeffreys [*Theory of probability*, Oxford Univ. Press, Oxford, 1939 and 1948; MR **1**, 151; section 1.6], which was designed to prove that if a sequence of experiments verify a hypothesis, then the probability tends to 1 that the next experiment will again verify the hypothesis. Popper claims that Jeffreys's conclusion is wrong and his claim is based on the fact that rival hypotheses could be invented consistent with all the observations to date. In my opinion these rivals would be of increasing complication and their probabilities would tend to zero, but Popper would reply, with remarkable originality, that all universal statements have precisely zero probability.

In order to try to prove this last statement he starts from an argument of H. Jeffreys and D. Wrinch [Philos. Mag. **42** (1921), 369–390] who decided that "a legitimate method must exist of assigning non-zero probabilities to an infinite sequence of explanatory theories". Popper believes that this conclusion would contradict the fact that "for a sufficiently large finite universe, the theory with the greater number of parameters will always be more probable (in the classical sense) than the theory with the smaller number of parameters" (pp. 380 and 384). His argument overlooks the fact that some values of a parameter (such as zero) may be more probable, initially, than others. He says, however, in a footnote (p. 384), "Jeffreys, loc. cit., remarks that 'half the prior probability [of a_2] is concentrated at $\alpha_{m+1}=0$,' which seems to mean that $p(a_1)=p(a_2)/2$; but this rule may lead to contradictions if the number of parameters of a_2 is greater than 2." (The meaning is clear enough.) But a self-consistent rule can be given for the initial (prior) probabilities of a hypothesis that contains a countably infinite number of parameters. Let $p_1, p_2, \cdots$ be such that $0<p_r<1$ $(r=1, 2, \cdots)$ and $\prod p_r>0$, and let $\alpha_r=p_r/(1-p_r)$. Then we could take the initial probability that the parameters of ordinal numbers $r_1, r_2, \cdots$ are zero as $\prod_r p_r/\prod_i \alpha_{r_i}$. (Only those sub-hypotheses containing an infinite number of non-zero parameters will then be ascribed zero probability.) This example refutes Popper's main argument against scientific induction, and we can all breathe again.

Popper wishes to replace the principle of scientific induction by a principle that scientists should formulate hypotheses and then allow these hypotheses to 'prove their mettle' in virtue of unsuccessful attempts to refute them. The justification of this recommendation is either inductive, which Popper would not allow, or it is itself a hypothesis which has to prove its mettle. But then we should again be in an infinite regress; in other words Hume's criticism of induction can be adapted to become an equally cogent criticism of Popperian mettle-proving.

To say that scientific induction is never very useful is implicitly equivalent to the assumption that the log-factor (weight of evidence, or logarithm of the simple likelihood ratio) obtained from experimental tests is always numerically much greater than the initial log-odds of a theory, and this assumption is false.

Another thing that Popper is against is subjective probability. His arguments on this subject have been largely superseded by his paper in Dialectica **11** (1957), 354–374, in which he claims that the subjective theory cannot lead to results which are compatible with the objective theory. This is because the knowledge of previous experimental results must influence the value of a subjective probability. This argument is answered by the observation that the numerical value of a physical probability is equal to the limiting value of a subjective probability when an experiment is repeated indefinitely under essentially constant circumstances.

In Appendix iv a method is given for constructing, in a sense, actual models of a random sequence (random as far as n-tuples are concerned). The author has kindly supplied me with previous relevant references on the subject, namely Jean Ville, *Étude critique de la notion de collectif* [Thesis, Paris, 1939], and A. H. Copeland, Amer. J. Math. **50**, 1928. These references were not known to Popper in 1934. Further papers, mathematically relevant, but not concerned with the theory of random sequences,

are I. J. Good, J. London Math. Soc. **21** (1946), 167–169 [MR 8, 430]; D. Rees, ibid. **21** (1946), 169–172 [MR 8, 431]; N. G. de Bruijn, Indag. Math. **8** (1946), 461–467 [MR 8, 247]; N. M. Korobov, Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 17–46 (=Amer. Math. Soc. Transl. (2) **4** (1956), 31–58) [MR **13**, 213]; R. B. Dawson and I. J. Good, Ann. Math. Statist. **28** (1957), 946–956 [MR **20**, 58]; M. H. Martin, Bull. Amer. Math. Soc. **40** (1934), 859–864.

In Appendices *ii to *v, Popper develops a formal theory of probability, largely in order to provide rigour to his argument that the degree of corroboration or acceptibility of a theory is not a probability. The reviewer is surprised that anyone ever supposed that it was, although Popper says (p. 394), "I regard the doctrine that degree of corroboration or acceptibility cannot be a probability as one of the most interesting findings of the philosophy of knowledge." The general notion of corroboration is surely used by other authors, even to the extent of giving it a name. For example, H. Jeffreys [Proc. Cambridge Philos. Soc. **32** (1936), 416–445] used the word 'support', and the reviewer the expression 'weight of evidence' [*Probability and the weighing of evidence*, Charles Griffin, London, 1950; MR **12**, 837; Chap. 6] for much the same thing.

In Popper's formal theory of probability he takes very seriously the possibility that 'given' propositions may have zero probability, primarily because he believes that universal statements have zero probability. (My theory also permits 'almost impossible' propositions to be given, but with less emphasis since I consider that the only application is to idealized mathematical situations.)

Popper's discussion of corroboration stimulated me to write on the subject [J. Roy. Statist. Soc. Ser. B, to be published]. His desiderata do not include the following condition. If evidence is considered in two parts, E and F, then the (degree of) corroboration of H is analytically determined by that provided by E together with that provided by F when E is known. An explicatum for $C(H:E|G)$, the corroboration of H provided by E, given G, that satisfies this condition and also all of Popper's (with some small modifications and elaborations) is any differentiable increasing function, g, of $f(P(H|E\cdot G))-f(P(H|G))$, where $f(p)$ is a strictly convex decreasing function of $-\log p$, which tends asymptotically to $\log p$ when $p\to 0$, and such that $k'(p)>(1-p)k''(p)$. (The explicatum is additive if $g(x)$ is the function x.) Permissible explicata are therefore $\lambda(P(H|E\cdot G)-P(H|G))+\mu I(H:E|G)$ $(\lambda>0,\ \mu>0)$, and $W(H:E|G)$, where $I(H:E|G)$ is $\log(P(E|H\cdot G)/P(E|G))$, the 'unexpectated' amount of information concerning H provided by E given G [I. J. Good, Proc. Inst. Elec. Engrs. C (3) **103** (1956), 200–204; MR **17**, 868], and $W(H:E|G)=\log(O(H|E\cdot G)/O(H|G))=\log(P(E|H\cdot G)/P(E|\bar{H}\cdot G))$, $=I(H:E|G)-I(\bar{H}:E|G)$, the weight of evidence for H provided by E given G.

The great importance that Popper rightly attributes to the ease of falsifiability of a theory flows readily from a subjectivistic philosophy. The subjectivist accepts hypotheses on the basis of their probabilities and utilities. One aspect of the utility of a theory, H, that is in competition with, say, an orthodox theory, $\bar{H}$, is the distinguishability between the two theories on the basis of evidence. This can be measured by the expected weights of evidence for H, given H and given $\bar{H}$, namely

$$\sum_i P(E_i|G)\log(P(E_i|H)/P(E_i|\bar{H}))$$

where $E_1, E_2, \cdots$ are the possible results of a large set of experiments, and G is H or $\bar{H}$. [Cf. I. J. Good, *Probability and the weighing of evidence*, p. 72; S. Kullback and R. A. Leibler, Ann. Math. Statist. **22** (1951), 79–86; MR **12**, 623.] This merit of distinguishability may be a matter of confirmability or falsifiability or both, but Popper believes that only falsifiability is important. More generally, expected (additive) corroborability may be used as a measure of utility, expected amount of information being a limiting case. [See D. V. Lindley, Ann. Math. Statist. **27** (1956), 986–1005; MR **18**, 783; where expected amount of information is used to measure the utility of an experimental design. He gives further references.] A metaphysical theory derives little utility from its expected corroborability, but it may make up for this by dealing with very important questions.

Popper (p. 142) identifies simplicity with testability and therefore presumably with falsifiability. But it is easy to construct complicated theories that are readily falsifiable, and I find myself in strong disagreement with Popper here. His use of the word "simple" conflicts too much with normal usage.

The book contains some very interesting observations about quantum theory. In particular it is shown decisively that a familiar interpretation of Heisenberg's uncertainty principle is untenable. The principle does not after all rule out the determination of the precise simultaneous position and momentum of a particle, provided that the determination is done retrospectively and not predictively. The argument is apparently correct even if particles do not exist.

Statements can be right and stimulating, right but not stimulating, wrong and not stimulating, or wrong but stimulating. Most of Popper's statements belong to the first and last of these classes. *I. J. Good* (Teddington)

6319:

★**Destouches, J. L. Physico-logical problems.** The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 390–405. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

This paper (1) surveys physico-logical problems and (2) summarizes results in this area. (1) Physico-logical problems concern the relations between formal structures and physical interpretations of theories. "... physico-logical studies can ... set the theoretical developments in their right connection with experiment." These problems are amenable to rigorous treatment by formal methods. (2) Results which exemplify but do not exhaust the area concern "the general theory of predictions", an abstract treatment of prediction independent of particular physical theories. This provides "a frame to discuss general properties of a physical theory".

W. Salmon (Providence, R.I.)

6320:

★**Braithwaite, R. B. Axiomatizing a scientific system by axioms in the form of identifications.** The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the

Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 429–442. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

A "Campbellian axiom" of a scientific system is one whose primitive non-logical terms are exclusively theoretical terms. A "dictionary axiom" is one whose primitive terms are both observable and theoretical and which suitably interrelate the two (perhaps in the form of identifying the observable property as a suitable function of the theoretic notions). Most axiomatizations of scientific theories contain Campbellian axioms among their non-logical axioms, but this is not always necessary. Indeed, the author shows that it is often possible to replace Campbellian by identificatory dictionary axioms, with no resulting change in the testable consequences, if the system is sufficiently simple. The author maintains that a system with no Campbellian axioms represents Eddington's ideal for physical theory as an identificatory system. But it is far from clear whether such a system of sufficient power could ever be realized in practice.

R. M. Martin (Bonn)

6321:

Robinson, Abraham. Outline of an introduction to mathematical logic. III, IV. Canad. Math. Bull. **1** (1958), 193–208; **2** (1959), 33–42.

These are the concluding installments of an exposition of elementary mathematical logic based on a course of fifteen lectures presented to the Edmonton (1957) Seminar of the Canadian Mathematical Congress; for the preceding installments see same Bull. **1** (1958), 41–54, 113–127 [MR **20** #5123, #5124]. The whole series forms a rather remarkable expository job. The author sets forth the principal features of the classical propositional calculus, the classical predicate calculus, with some remarks about Boolean algebra. The treatment is intended for mature mathematicians—the author uses Zorn's lemma, and assumes the reader is familiar with it—but requires no previous acquaintance with mathematical logic. The preceding installments treated classical propositional calculus and ended with a sketch of Boolean algebra and its relations to the propositional calculus. The present installments continue the discussion of Boolean algebra with particular reference to ideals (actually dual ideals or filters). The author then introduces the predicate calculus, discusses it first semantically and then deductively, and ends with a proof of the Gödel completeness theorem. There are quite a number of misprints; if a reader finds a passage difficult to follow he should experiment with leaving out or putting in an extra negation sign, interchanging the null set and the universal set, and similar decoding transformations, and he will doubtless find that what the author meant is in reality quite clear and simple.

H. B. Curry (University Park, Pa.)

6322:

Kamiński, Stanisław. On the number of concluding syllogistic moods. Studia Logica **8** (1958), 165–176. (Polish. Russian and English summaries)

The author gives a taxonomy of classical logical forms.

N. S. Mendelsohn (Winnipeg, Man.)

6323:

Głazowska, Krystyna. The structure of valid n-term syllogisms. Studia Logica **8** (1958), 249–257. (Polish. Russian and English summaries)

The author defines an n-syllogism as a sequence of n categorical sentences every consecutive pair of which has a common term. She gives a set of rules for deciding validity.

N. S. Mendelsohn (Winnipeg, Man.)

6324:

Cuninghame-Green, R. A. Single primitive ternary connectives for the 2-valued propositional calculus. Z. Math. Logik Grundlagen Math. **5** (1959), 206–207.

A report on those ternary connectives each of which generates (by iteration) all truth-tables for the (2-valued) propositional calculus.

B. A. Galler (Ann Arbor, Mich.)

6325:

Orey, Steven. Model theory for the higher order predicate calculus. Trans. Amer. Math. Soc. **92** (1959), 72–84.

The predicate calculus of order ω (PC^ω) considered in this paper is formulated for sets of higher types (so that functions are introduced by representing sets). 'Model' is understood in the sense of Henkin [J. Symb. Logic **15** (1950), 81–91; MR **12**, 70]. With each domain D of individuals is associated the maximal or standard model $\mathbf{D}$ (of PC^ω) consisting of all subsets 2^D, subsets of 2^D, $2^D \times \cdots \times 2^D$, etc. Each model G (of a set of sentences of PC^ω with the domain G_0 of individuals) $\subseteq \mathbf{G}_0$ ('$\subseteq$' in the set theoretic sense). A partial ordering $G < G'$ is defined to hold just in case $G_0 = G_0'$ and $G \subseteq G'$ ($\subseteq \mathbf{G}_0$). The author is interested in syntactic and model theoretic properties of formulae ϕ (of PC^ω) which satisfy one of the two conditions and their converses: (i) If ϕ is true in G, then also in $\mathbf{G}_0$; and (ii) if ϕ is true in G and $G < G'$, then ϕ is true in G'. A well-known example of (ii) is any prenex ϕ all of whose higher type quantifiers are existential. The author gets syntactic descriptions of larger classes of formulae satisfying (i) and (ii) respectively and hence, in terms of the—undecidable—relations of formal equivalence and equivalence in standard models, a characterization of all such formulae. Further, he relativises the results (considering not all models of PC^ω but only those G_H which satisfy a set of sentences $\mathscr{H}$) and obtains analogues, for ϕ satisfying (i) or (ii), to the compactness of PC^1, at least as long as $\mathscr{H}$ ensures that, for each relevant type, the set of all elements of type τ belonging to G_H is itself a member of type (τ) of G_H. It may be remarked that Kleene's characterization of hyperarithmetic sets (HA) fits into the framework of the present paper: each HA is definable by a formula $(X)A(X, n)$, where $(*)$ $(X_0)A(X_0, n) \leftrightarrow (EX_0)B(X_0, n)$, A and B arithmetic, X a set variable, X_0 ranging over the maximal model over N. The converse to (ii) holds, with $(X)A(X, 0^{(n)})$ for ϕ, $G_{(*)}$ for G, and $G_{(*)}'$ for G'.

G. Kreisel (Reading)

6326:

Evans, Trevor; and Schwartz, P. B. On Słupecki T-functions. J. Symb. Logic **23** (1958), 267–270.

Let L_t be the logic obtained from the m-valued Łukasiewicz-Tarski logic by adding the unary connective T_t such

that $T_i(p)$ always has truth-value i. The main result is that L_t is functionally complete if and only if $m-1$ and i_0-1 are relatively prime, where $i_0=\min(i, m-i+1)$.

E. Mendelson (New York, N.Y.)

6327:

Gavrilov, G. P. Certain conditions for completeness in countable-valued logic. Dokl. Akad. Nauk SSSR **128** (1959), 21–24. (Russian)

Let P be the set of functions with integral arguments and values. A subset N of P is called complete if its closure with respect to superposition [cf. S. V. Yablonskiĭ, Trudy Mat. Inst. Steklov. **51** (1958), 5–142; MR **21** #3331] is P. N is called precomplete if it is not complete but addition to it of any other function in P yields a complete set. This paper describes a family of precomplete classes and a special precomplete class, and gives two criteria of completeness for subsets N of a special form.

E. Mendelson (New York, N.Y.)

6328:

★Specker, Ernst. Der Satz vom Maximum in der rekursiven Analysis. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 254–265. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

An example of a (continuous) recursive real function which does not achieve its maximal value in the interval $[0, 1]$ at a recursive real number. A function f from rationals to rationals with the following properties is constructed: (1) $0\leq f(x)\leq 1/36$. (2) For $x\notin[0, 1]$, $f(x)=0$. (3) $|f(x)-f(x')|\leq|x-x'|$. (4) For every $0<\varepsilon$ there exists an x such that $1/36-\varepsilon\leq f(x)$. (5) f is primitive recursive. (6) For every recursive and recursively convergent sequence of rationals $u(n)$ we have $\lim_{n\to\infty} f(u(n))<1/36$. The continuous completion of f to a function from reals to reals is the desired example. The construction of f depends on the existence of an infinite recursive tree (with at most two branches at each node) which does not possess any infinite recursive path [see also Kleene, Proc. Internat. Congr. Math., Cambridge, Mass., 1950, vol. 1, pp. 679–685, Amer. Math. Soc., Providence, R.I., 1952; MR **13**, 422].

M. O. Rabin (Jerusalem)

6329:

★Davis, Martin. Computable functionals of arbitrary finite type. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 281–284. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

The author proposes a definition of computability of functionals. The functional u having empty domain and range is of type 0. A functional is of type $n+1$ if its domain consists of functionals of type n and its range is a set of natural numbers (ramified types and function-valued functionals are also considered). Two functionals are compatible if they possess a common extension. u is consistent; f is consistent if its domain consists of consistent functionals and for every pair of compatible functionals w and v in its domain $f(w)=f(v)$. u is finite; f is finite if its domain consists of a finite number of finite functionals. Let $\langle f\rangle$ be a Gödel numbering of finite functionals. A (consistent) functional f is completely computable (c.c.) if there exists a partial recursive function q such that $f(v)=k$ if and only if, for some finite functional w, v is an extension of w and $q(\langle w\rangle)=k$. Normal form theorems, similar to those for recursive functions, hold for c.c. functionals. A notion of computable functionals is introduced. For type $n\geq 4$ there are no computable functionals.

M. O. Rabin (Jerusalem)

6330:

Oberschelp, Arnold. Über die Axiome produkt-abgeschlossener arithmetischer Klassen. Arch. Math. Logik Grundlagenforsch. **4** (1958), 95–123.

A 'product axiom' is a first order sentence that holds true in any direct product of (two) models (for example, algebraic systems) whenever it is true in each factor. A first and basic result (Satz 1) is that the set Π of all product axioms of a given language with finitely many primitives is recursively enumerable. {That Π is not recursive has been announced by Feferman [Notices Amer. Math. Soc. **6** (1959), 619–620]; however, a result of Craig [J. Symb. Logic **18** (1953), 30–32; MR **14**, 1051] implies that Π as a recursively enumerable set possesses a recursive subset Π' such that each sentence in Π is equivalent to some sentence in Π', whence, in formal analogy with the case of other algebraic constructions, a property is preserved under products if and only if it is expressible by a sentence from the recursive set Π'.} This is established by associating with each sentence S a new sentence $\mathrm{Pr}(S)$, in a language obtained by 'doubling' the non-logical constants of the language containing S, which expresses in a natural way that if the sentence S holds in each factor, then it holds in the product; thus S is in Π if and only if $\mathrm{Pr}(S)$ is logically derivable. If Σ is the set of all universal or existential sentences, then the sentences in $\mathrm{Pr}(\Sigma)$ can all be written in universal-existential form, and it follows from a result of Ackermann [cited in the paper under review] that $\mathrm{Pr}(\Sigma)$ and hence $\Pi\cap\Sigma$ are decidable. Satz 8 provides a criterion similar to that of Satz 1, now with $\mathrm{Pr}'(S)$ in the same language as S, but of exponential length. Further considerations concern syntactical classes Σ such that the elements of $\Pi\cap\Sigma$ are equivalent to Horn sentences, and include cited results of McKinsey, of Horn, and of Bing; a final section treats powers, that is, products of isomorphic factors.

R. C. Lyndon (Princeton, N.J.)

6331:

Lyndon, Roger C. Properties preserved in subdirect products. Pacific J. Math. **9** (1959), 155–164.

Formulas in a first-order language (with symbols for the operations and relations, including identity) are considered. The author defines the concepts: atomic formula (obtained by inserting terms in the argument places of a relation symbol), positive formula (constructed from atomic formulas by means of conjunction, disjunction and quantification), special Horn formula (this is a formula obtained by conjunction and universal quantification from the formulas of the form $P\supset F$, where P is a positive formula and F is an atomic formula). A sentence is a formula without free variables. The author proves the following theorem. Let R be a relational system of the language L, and K an elementary class of systems of L. Then (1) and (2) are equivalent: (1) R satisfies all special

Horn sentences that hold in K; (2) R has an elementary extension that is a subdirect product of systems in K.

Corollary: A sentence has the property that it holds for a subdirect product of systems whenever it holds for each component system if and only if it is equivalent to a special Horn sentence. *J. Jakubík* (Košice)

SET THEORY

6332:

de Bruijn, N. G. Addendum to "A theorem on choice functions". Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 327.

An error in the proof of a result in same Proc. **60** (1957), 409–411 [MR **19**, 1152] is corrected.

S. Ginsburg (Gardena, Calif.)

COMBINATORIAL ANALYSIS

See also 6389, 6579, 6950.

6333:

Goldberg, K. Principal submatrices of a full-rowed non-negative matrix. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 19–20.

The main theorem proved in this paper states that, if every element of a square matrix is 0 or 1 and if no row consists entirely of zeros, then at least one principal submatrix is a permutation matrix. Some deductions are made from this result and it is shown, in particular, that each principal submatrix of a nilpotent non-negative matrix is nilpotent and possesses both zero rows and zero columns. This, in turn, leads to an extremely simple algorithm for deciding whether a given non-negative matrix is nilpotent. Finally, it is pointed out that the main theorem is equivalent to a statement concerning directed graphs and that in this form it had been proved in essence, by D. Rosenblatt [Naval Res. Logist. Quart. **4** (1957), 151–167; MR **19**, 889]. *L. Mirsky* (Sheffield)

6334:

Dijkstra, E. W. A note on two problems in connexion with graphs. Numer. Math. **1** (1959), 269–271.

Algorithms are given for (1) finding the shortest spanning tree for a graph, (2) finding the shortest path connecting two points of a graph, which are claimed to be more economical computationally than those previously published. *D. Gale* (Providence, R.I.)

6335:

Goodman, A. W. On sets of acquaintances and strangers at any party. Amer. Math. Monthly **66** (1959), 778–783.

Let each of the $N(N-1)/2$ edges of the full graph with N vertices be colored with one of two colors. Then each of the $N(N-1)(N-2)/6$ triangles will be either monochromatic or bichromatic. The author obtains the precise lower bound in terms of N for the number of monochromatic triangles that may appear.

A. M. Gleason (Cambridge, Mass.)

ORDER, LATTICES

See also 6375, 6401.

6336:

Gravett, K. A. H. "Netrics" of sets and linear spaces. Quart. J. Math. Oxford Ser. (2) **10** (1959), 9–16.

Let Δ be a partially ordered set and let T be the set of all trivially ordered subsets of Δ including the null set $\emptyset$. If $\delta \in \Delta$ and $s \subseteq \Delta$, then $\delta \not\leq s$ means that no element of s is greater than or equal to δ. A pair $[E, d]$ is a netric set if E is a set and d is a mapping of $E \times E$ into T so that, for x, y, z in E and δ in Δ, (i) $d(x, y) = \emptyset$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$; if $\delta \not\leq d(x, y) \cup d(y, z)$, then $\delta \not\leq d(x, z)$. A netric set $[E_1, d_1]$ is an immediate extension of a netric set $[E_2, d_2]$ if $E_1 \supseteq E_2$, $d_2 = d_1 | E_2 \times E_2$ and for each y in $E_1 \backslash E_2$, x in E_2 and δ in $d(x, y)$ there corresponds x' in E_2 with $\delta \not\leq d(x', y)$. A netric set is complete if it does not admit a proper immediate extension. For a certain class of "equi-cardinal" netric sets it is shown that there exist unique immediate complete extensions. Then a systematic study of the structure of complete netric sets is made. Let L be a linear space over the field K. A mapping d of L into T is a netric of L if, for each x, y in L, for k in K^*, and δ in Δ, (i) $d(x) = \emptyset$ if and only if $x = 0$; (ii) $d(kx) = d(x)$; (iii) if $\delta \not\leq d(x) \cup d(y)$, then $\delta \not\leq d(x+y)$. For x, y in L put $d^*(x, y) = d(x-y)$; d^* is a netric of the set L. There is a natural 1-1 correspondence between netrics of the linear space L and an easily characterized set of equi-cardinal netrics of the set L. Thus the results obtained for netric sets apply to netrics of linear spaces. These results generalize those of the reviewer [Amer. J. Math. **75** (1953), 1–29; MR **14**, 842].

P. F. Conrad (New Orleans, La.)

6337:

Jakubík, Ján. Vertauschbare Kongruenzen in Verbänden. Mat.-Fyz. Časopis. Slovensk. Akad. Vied. **8** (1958), 155–162. (Slovak. Russian and German summaries)

In der vorliegenden Arbeit werden Kongruenzen in Verbänden bezüglich ihrer gegenseitigen Vertauschbarkeit untersucht. Verf. zeigt zunächst, dass ein Satz von H. A. Thurston [Proc. Edinburgh Math. Soc. (2) **10** (1954), 76–77; MR **16**, 559], nach dem in einem distributiven Verband mit gewissen Endlichkeitseigenschaften je zwei Kongruenzen vertauschbar seien, nicht zutrifft.—Sei S ein Verband, R eine Kongruenz in S; für $x \in S$ bezeichnen wir mit $\bar{x}$ die das Element x enthaltende Klasse von S/R. Zwei Kongruenzen R_1, R_2 in S sind nur dann vertauschbar, wenn (K_1) für x, u, $v \in S$, $u < x < v$, $u \equiv x$ (R_1), $x \equiv v$ (R_2), im Verband S/R $(R = R_1 \cap R_2)$ ein relatives Komplement von $\bar{x}$ in dem Intervall $\langle \bar{u}, \bar{v} \rangle$ existiert. Sei (K_2) die Bedingung, die aus (K_1) durch Vertauschung der Indizes 1, 2 entsteht, und (K) diejenige, die das gleichzeitige Bestehen von (K_1) und (K_2) ausdrückt. Es wird gezeigt, dass die Bedingung (K) für die Vertauschbarkeit der Kongruenzen R_1, R_2 notwendig und hinreichend ist. Ausserdem werden zwei andere Sätze ähnlicher Natur bewiesen. *O. Borůvka* (Brno)

6338:

Pierce, R. S. Translation lattices. Mem. Amer. Math. Soc. no. 32, 66 pp. (1959).

The paper is connected with the work of R. P. Dilworth

[Trans. Amer. Math. Soc. **68** (1950), 427–438; MR **11**, 647] and I. Kaplansky [Amer. J. Math. **70** (1948), 626–634; MR **10**, 127]. It is devoted to a canonical representation theory of abstract translation semi-lattices. A semi-lattice T (the binary operation is denoted by $\wedge$) is called a translation semi-lattice (briefly t.s.l.) if for any real number α an automorphism $f \to f+\alpha$ of T exists such that: (1) $f+0=f$; (2) $(f+\alpha)+\beta=f+(\alpha+\beta)$; (3) $\alpha>0 \Rightarrow f+\alpha>f$; (4) for $f, g \in T$ an α exists such that $f+\alpha \geqq g$; (5) $(f \leqq g+\alpha$ for all $\alpha>0) \Rightarrow f \leqq g$. On a t.s.l., the uniform topology is defined by the distance function $\rho(f,g)=\inf\{\lambda \mid f-\lambda \leqq g \leqq f+\lambda\}$. The homomorphisms of t.s.l., the factor t.s.l. $T/\mathfrak{I}$ (an ideal $\mathfrak{I}$ is a non-empty set such that $f \in \mathfrak{I}$, $g \in T$, $g \leqq f \Rightarrow g \in \mathfrak{I}$) and the notion of a divisible t.s.l. with respect to an ideal are considered. The ground for the first representation theorem is the notion of a lattice function F on a complete Boolean algebra B. If z is the zero element of B, then F is a bounded real-valued function on $B-\{z\}$ such that $F(a) \leqq F(b)$ whenever $z \neq b \leqq a$. A lattice function F is called normal if the equality $F(\sup A)=\inf\{F(a) \mid a \in A\}$ holds for any $A \neq \emptyset$, $A \subseteq B-\{z\}$. The set $N(B)$ of all normal functions on B is a t.s.l. Let $S \neq \emptyset$ be a sub-t.s.l. of $N(B)$ closed under the translation operation, $Z(S)=\{F \in S \mid F(a) \leqq 0$ for all $a \in B-\{z\}\}$. The first main theorem: Let T be a t.s.l. and $\mathfrak{I}$ a closed ideal of T such that $T/\mathfrak{I}$ has a unit element. For the pair $(T, \mathfrak{I})$ there exists a uniquely determined triple (B, S, Θ), where (1) B is a complete Boolean algebra, (2) S is a sub-t.s.l. of $N(B)$ for which $\{a(F) \mid F \in S\}$ is dense in B $(a(F)=\sup\{b \in B-\{z\} \mid F(b)>0\})$, (3) Θ is a homomorphism of T onto S such that $\Theta^{-1}(Z(S))=\mathfrak{I}$.

The second representation theorem deals with the canonical representation of t.s.l. by continuous functions: For the pair $(T, \mathfrak{I})$ there exists a uniquely determined triple (X, C, Γ), where (1) X is a compact Hausdorff space, (2) C is a sub-t.s.l. of the t.s.l. of the continuous functions on X which separates the points of X, and for any open set $\emptyset \neq N \subseteq X$ there exists $f \in C$ such that $\emptyset \neq \{x \in X \mid f(x)>0\} \subseteq N$, (3) Γ is a homomorphism of T onto C such that $\Gamma^{-1}(Z(C))=\mathfrak{I}$ (where $Z(C)=\{f \in C \mid f \leqq 0\}$).

The author gives numerous examples and applications of the abstract results to the investigation of t.s.l. of functions. *V. Vilhelm* (Prague)

6339:

Matsushima, Yatarō. On extreme elements in lattices. Proc. Japan Acad. **35** (1959), 226–231.

In this paper the author introduces the concept of extreme elements in a lattice in terms of B-covers and B^*-covers, introduced in his earlier work [same Proc. **32** (1956), 549–553; **33** (1957), 328–332, 462–467; MR **18**, 713; **20** #2291, #2292]. Let L be a lattice and let $a, b, c, \cdots, x, y, \cdots$ denote elements of L. $B(a,b)=\{x \mid axb\}$ and $B^*(a,b)=\{y \mid aby\}$, where axb means that $x=(a \cup x) \cap (b \cup x)=(a \cap x) \cup (b \cap x)$. An element $e \in L$ is extreme to x if $B^*(x,e)=e$. e is extreme if e is extreme to some element of L. A pair (a,b) is an extreme pair if a is extreme to b and b is extreme to a. $E(a)$ denotes the set of all elements which are extreme to a. Typical results are the following. If a and a' are complemented then (a,a') is an extreme pair. In a compact lattice $E(a)=b$ implies $a \cup b=1$ and $a \cap b=0$. A compact lattice which is an extreme lattice is a complemented lattice (a lattice is extreme if all of its elements are extreme). *Ph. Dwinger* (Lafayette, Ind.)

6340:

Matsushima, Yatarō. Between-topology on a distributive lattice. Proc. Japan Acad. **35** (1959), 221–225.

[For notations and references see review above.] The author defines the B-topology [B^*-topology] on L by taking the sets $B(a,b)$ [$B^*(a,b)$] as a sub-basis of closed sets. It is proved that if L is a distributive lattice with O and I, then the B-topology is the interval topology. A distributive lattice L has the property (A) if the following holds: For any subset $B(a,b)$ in L, if $x, y \in B(a,b)$, then $a \cap x$ and $a \cap y$; $b \cap x$ and $b \cap y$ are comparable, respectively. L has the property (B) if L consists of a finite number of maximal extreme B-covers and a chain. The following results are obtained. If a distributive lattice L satisfies (A) then L is a topological lattice in the B^*-topology. If, moreover, L satisfies (B) and if L has an O and an I, then L is a Hausdorff space in its interval topology. This result is related to a theorem obtained by Wolk [Proc. Amer. Math. Soc. **9** (1958), 524–529; MR **20** #3079]. Finally it is shown that a distributive lattice L with O, I such that $L=B(a_0,b_0)$ is a mob with the desired kernel $B(a,b)$ and with the multiplication defined as follows: $xy=(a \cup x) \cap (b \cup y)$ for the fixed two elements a, b of L. *Ph. Dwinger* (Lafayette, Ind.)

6341:

Funayama, Nenosuke. Imbedding infinitely distributive lattices completely isomorphically into Boolean algebras. Nagoya Math. J. **15** (1959), 71–81.

It is well known that every distributive lattice can be imbedded in a Boolean algebra. This imbedding is in general not a complete isomorphic one, i.e., an isomorphic imbedding preserving infinite sums. It is obvious that in order that the imbedding be a complete isomorphic one, it is necessary that the lattice be infinitely distributive. In the paper under review, it is proved as a main result that this condition is also sufficient. The author also shows that every relatively complemented distributive lattice is infinitely distributive. Therefore such a lattice can also be imbedded completely isomorphically in a Boolean algebra. The author uses this result to generalize results obtained by E. C. Smith, Jr. and Tarski [Trans. Amer. Math. Soc. **84** (1957), 230–257; MR **18**, 865] on higher degrees of distributivity of Boolean algebras to the case of relatively complemented distributive lattices. A typical result is the following: Every relatively complemented $(\alpha, 2)$-distributive lattice is (α, α)-distributive.

Ph. Dwinger (Lafayette, Ind.)

6342:

Jordan, Pascual. Über distributive Schrägverbände. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1958**, 227–257.

Previous results [e.g. the author, Abh. Math. Sem. Univ. Hamburg **21** (1957), 127–138; MR **19**, 524; see also S. Matsushita, Math. Ann. **137** (1959), 1–8; MR **20** #6988] on skew-lattices and skew-semilattices are reformulated and extended. A $\wedge$-skew-semilattice H is called "smooth" if $a \wedge b \wedge a=a \wedge b$ for all $a, b \in H$, called "stretched" if $a \wedge b=a$ or $a \wedge b=b$, and called a "semi-nest" if $a \wedge b=a$. Emphasis is on the interrelationship of these and other subcategories, especially various forms of distributivity. Free smooth skew-semilattices are characterized. The free

smooth skew-lattice on two generators satisfying $a \vee (b \wedge c) = (a \vee b) \wedge c$ ("supermodularity") and $a \wedge [(a \wedge b) \vee b] = a \wedge b$ and dually has 18 elements.

P. M. Whitman (Silver Spring, Md.)

6343:

Halperin, Israel. The continuous geometries of J. von Neumann. Mat. Lapok **9** (1958), 225–231. (Hungarian)

This note is a summary of von Neumann's work on the theory of continuous geometries together with references to the subsequent development of this theory. Some as yet unpublished papers of von Neumann (from 1936–37, remaining as manuscripts) are also discussed.

G. Szász (Szeged)

6344:

Jakubík, Ján. Über Ketten in Booleschen Verbänden. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 193–202. (Slovak. Russian and German summaries)

Sei S ein Verband. Eine Kette $R \subset S$ heisst maximal in S, wenn für jede Kette $R_1 \subset S$, aus $R \subset R_1$ die Gleichheit $R = R_1$ folgt. Die vorliegende Arbeit enthält Untersuchungen über maximale Ketten im Zusammenhang mit der Jordan-Dedekindschen Bedingung. Sei S_0 der von allen Untermengen einer abzählbaren Menge A gebildete Boolesche Verband, J das Ideal aller endlichen Untermengen von A, $S = S_0/J$. Ferner sei R eine beliebige maximale Kette in S und k ihre Kardinalzahl. Es wird gezeigt, dass für jede Mächtigkeit α, $\alpha \geq k$, ein Boolescher Verband S_α existiert mit der folgenden Eigenschaft: Zu jeder Mächtigkeit β, $k \leq \beta \leq \alpha$, gibt es in S_α eine maximale Kette R_β mit der Kardinalzahl β. Ausserdem werden einige weitere Resultate ähnlicher Art abgeleitet.

O. Borůvka (Brno)

6345:

Itoh, Makoto. On Boolean equation with many known elements and generalized Poretzky's formula. Univ. Nac. Tucumán. Rev. Ser. A **12**, 107–112 (1959).

The standard results concerning the solution of a Boolean equation with one unknown are generalized to cover the case of a Boolean equation with n unknowns.

A. A. Grau (Oak Ridge, Tenn.)

GENERAL MATHEMATICAL SYSTEMS

See 6330, 6331.

CLASSICAL ALGEBRA

6346:

Barsotti, Leo. Condition for the roots of an equation of fourth degree to constitute a harmonic group. Soc. Parana. Mat. Anuário (2) **1** (1958), 35–37. (Portuguese. French summary)

Let (x, y) be the homogeneous coordinates of points on a line; the author shows that the four roots of

$$Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4 = 0$$

form a harmonic range if

$$\begin{vmatrix} A & B & C \\ B & C & D \\ C & D & E \end{vmatrix} = 0.$$

R. G. Stanton (Waterloo, Ont.)

THEORY OF NUMBERS

See also 6366, 6411, 6568.

6347:

Golomb, Solomon W. A connected topology for the integers. Amer. Math. Monthly **66** (1959), 663–665.

In the set N of all positive integers we consider the arithmetic progressions $\{an+b\}$ with $(a, b) = 1$ (restricted to those integers n for which $an + b \in N$). A topology D for N is introduced by taking the collection of all these progressions as a basis for the open sets. It is shown that this topology is Hausdorff and connected, but not regular (the point 1 and the set of all even numbers do not have disjoint open neighborhoods). And Dirichlet's theorem on primes in arithmetic progressions is equivalent to the statement that the set of all primes is everywhere dense in N. There is a further theorem on a certain topology D'; it is stated that D' is stronger than D, but it should be remarked that $D' = D$, as the open sets of D' arise from the open sets of D by omitting finite sets.

N. G. de Bruijn (Amsterdam)

6348:

Erdös, P. Remarks on number theory. II. Some problems on the σ function. Acta Arith. **5** (1959), 171–177.

Let $\sigma(n) = \sum_{d|n} d$. The author proves that the number of distinct numbers of the form $\sigma(n)/n$, $1 \leq n \leq x$, equals $c_1 x + o(x)$, where $6/\pi^2 < c_1 < 1$, and gives the outline of a proof that the number of solutions of the equation $\sigma(n)/n = \sigma(m)/m$, $m < n \leq x$, equals $c_2 x + o(x)$, for some positive constant c_2. The arguments are elementary and ingenious. Some interesting conjectures are described, and references to related work given [see particularly B. Hornfeck und E. Wirsing, Math. Ann. **133** (1957), 431–438; MR **19**, 837].

H. Halberstam (London)

6349:

Cohen, Eckford. An arithmetical inversion principle. Bull. Amer. Math. Soc. **65** (1959), 335–336.

An arithmetical function $f(n, r)$ of two variables is said to be an even function of n (mod r) if $f(n, r) = f((n, r), r)$ for all integers n and all positive integers r. The author announces an inversion relation for even functions which may be described as follows: If $r = r_1 r_2$, then

$$g(r_1, r_2) = \sum_{d|r_1} f(r_1/d, r)\eta(d) \rightleftarrows \sum_{d\delta=(n,r)} g(d, r/d)\xi(\delta) = f(n, r),$$

where ξ and η are two arithmetical functions such that $\sum_{d|r} \xi(d)\eta(r/d) = [1/r]$. Interesting special cases occur when $\xi(r) = 1$ and $\eta(r) = \mu(r)$ or when $\xi(r) = \mu(r)$ and $\eta(r) = 1$. Applications are indicated.

T. M. Apostol (Pasadena, Calif.)

6350:

Carlitz, L. Kummer's congruences (mod 2^r). Monatsh. Math. **63** (1959), 394–400.

The author proves the following result, stated by Frobenius [S.-B. Preuss. Akad. Wiss. **1910**, 809–847; p. 845]: If a, b, r are integers, $a \geq 0$, $b > 0$, $r > 0$, then 2 divides both $\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} E_{2a+2bs}$ and $(2b)^r \cdot r!$ to the same power; here E_n is the coefficient of $t^n/n!$ in the expansion of $2e^t/(e^t+1)$. This result is complementary to known divisibility properties of a similar nature for odd primes. The author proves related results for Bernoulli numbers and others. *H. Halberstam* (London)

6351:

Katz, Alexander. Primes in the 13-th million. Riveon Lematematika **13** (1959), 32–34. (Hebrew and English)

A list is given of 1,831 primes from 12,012,017 to 12,041,099 inclusive, as an extract of a table of factorizations over approximately the same range. No explanation is given for the range or use of hand labor.

H. Cohn (Tucson, Ariz.)

6352:

Cohn, Harvey. Numerical study of the representation of a totally positive quadratic integer as the sum of quadratic integral squares. Numer. Math. **1** (1959), 121–134.

This paper describes almost 200,000 experiments in representing real quadratic integers as sums of squares of integers in the same quadratic field. For such a representation to be possible the number being represented must be totally positive. Results indicate that if such a number is not the sum of 5 or fewer squares then it is not the sum of any number of squares. (The smallest totally positive quadratic integer which is not a sum of squares in its own field is $6+\sqrt{6}$.)

To give further details, let m be a positive square-free rational integer and let $f=2$ or 1 according as $m \equiv 1 \pmod 4$ or not. The numbers $\alpha=(fa+2b\sqrt{m})/f^2$, where a and b are rational integers such that $a \equiv b \pmod f$, $a>0$, $b \geq 0$, $fa > 2b\sqrt{m}$, are the only integers in $K(\sqrt{m})$ that have a chance to be sums of squares in $K(\sqrt{m})$. For a given m a large number of suitable pairs $[a, b]$ were selected and the corresponding α's represented by the sum of the least number Q of squares. Tallies were kept of the numbers Q and the results tabulated for each m. The most popular number in almost all cases was $Q=3$. The 50 numbers m considered include all the square-free integers ≤ 41, three triples 86, 87, 89; 173, 174, 177; 255, 257, 258; two larger numbers 1487, 3467 and thirteen small integers with square factors. The calculations were done by GEORGE, the Argonne Laboratory's computer, by a typically elaborate program complicated by the fact that in $K(\sqrt{m})$ there are infinitely many units. The reader must consult the paper for further details.

D. H. Lehmer (Berkeley, Calif.)

6353:

Chen, Jing-jun. Waring's problem for $g(5)$. Sci. Record (N.S.) **3** (1959), 327–330.

If $g(k)$ denotes the least g such that every positive integer is representable as a sum of at most g positive kth powers, then using a method similar to that of Hua, it is shown here that $37 \leq g(5) \leq 40$. Dickson [Bull. Amer. Math. Soc. **39** (1933), 701–727] had shown that $37 \leq g(5) \leq 54$.

H. Gupta (Chandigarh)

6354:

Newman, Morris. Construction and application of a class of modular functions. II. Proc. London Math. Soc. (3) **9** (1959), 373–387.

The author continues the development of his earlier paper [same Proc. **7** (1957), 334–350; MR **19**, 953], concerning functions defined on $\Gamma_0(n)$, where, now, n need not be prime to 6. The author gives integral polynomial bases on F_n (the entire modular functions on $\Gamma_0(n)$) and on E_n (those F_n with negative valence only at $i\infty$) for $n=6$, 10, 14, 15, 21, 22, 26. Typically the bases of E_6 and F_6 are $\{a_1\}$ and $\{a_1, a_3^{-1}\}$, respectively, where

$$a_1 - 5 = \eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)^5,$$
$$a_3 = \eta(2\tau)^{10}\eta(3\tau)^{14}/\eta(\tau)^2\eta(6\tau)^{22}.$$

These functions are supplied, as before, from the author's theorem I (now extended to include all n regardless of primality to 6): If δ denotes all divisors of n, $\delta\delta' = n$, and if the integers r_δ are so defined that, with $r_1 = 0$, the two sums $\sum (\delta - 1)r_\delta/24$ and $\sum (\delta' - n)r_\delta/24$ are integers and $\prod \delta^{r_\delta}$ is a perfect square, then it follows that $\prod \phi_\delta^{r_\delta} = g(\tau)$ is a function on $\Gamma_0(n)$, where $\phi_\delta = \eta(\delta\tau)/\eta(\tau)$. These $g(\tau)$ come in sufficient abundance to provide modular identities for $\eta(\tau)$ on different levels. The study of the expansions at $i\infty$ was aided this time by the IBM 704.

H. Cohn (Tucson, Ariz.)

6355:

Staś, W. Über eine Anwendung der Methode von Turán, auf die Theorie des Restgliedes im Primidealsatz. Acta Arith. **5** (1959), 179–195.

If we know a zero $\beta + i\gamma$ of the Riemann zeta function, then an explicit lower estimation for the error term in the prime number theorem can be given [P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688; Satz 30]. The author gives a similar result for the prime ideal theorem in connection with a root of the Dedekind zeta function. His result is, when specialized to the rational field, somewhat sharper than Turán's: he has

$$T^\beta \exp\{-3(\log T)(\log\log\log T)(\log\log T)^{-1}\}$$

where Turán had

$$T^\beta \exp\{-21(\log T)(\log\log T)^{-1/2}\}.$$

N. G. de Bruijn (Amsterdam)

6356:

Erdős, P.; and Rényi, A. Some further statistical properties of the digits in Cantor's series. Acta Math. Acad. Sci. Hungar. **10** (1959), 21–29. (Russian summary, unbound insert)

Let $q_1, q_2, \cdots$ be a fixed sequence of positive integers, with $q_k > 2$ for all k. In an earlier paper [Ann. Univ. Sci. Budapest Sect. Math. **1** (1958), 7–32; MR **21** #1288] the authors, together with P. Szüsz, considered the probabilistic behavior of the "digits" $e_n(x)$ in the Cantor expansion

$$x = \sum_{n=1}^{\infty} \frac{e_n(x)}{q_1 q_2 \cdots q_n}$$

where, for each n, $\varepsilon_n(x)$ is an integer with $0 \leq \varepsilon_n(x) < q_n$. In particular, they generalized a classical theorem of Borel by showing that if

$$f_n(k, x) = \sum_{\substack{1 \leq j \leq n \\ \varepsilon_j(x) = k}} 1, \quad Q_n(k) = \sum_{\substack{1 < j < n \\ q_j > k}} \frac{1}{q_j},$$

then, for all integers $k \geq 0$ for which $\lim_{n\to\infty} Q_n(k) = \infty$,

$$\lim_{n\to\infty} \frac{f_n(k, x)}{Q_n(k)} = 1 \quad \text{for almost all } x.$$

In the present paper they consider the behavior of $M_n(x) = \max_k f_n(k, x)$, the most frequent number among the first n digits. Put $Q_n = Q_n(1)$. (I) If $\lim_n Q_n/\log n = \infty$, then for a.a. x, $\lim_{n\to\infty} M_n(x)/Q_n = 1$. (II) If $0 < c_2 \leq q_n/n \leq c_3$ for $n = 1, 2, \cdots$, and $\lim Q_n/\log n = \alpha > 0$, then, for a.a. x, $\lim_{n\to\infty} M_n(x)/Q_n = y(\alpha)$, where $y(\alpha)$ is the real solution of the equation $y \log y = \alpha^{-1}$. (III) If $q_n/n \to \infty$ but also $Q_n \to \infty$ as $n \to \infty$, then, for a.a. x, $\lim_{n\to\infty} M_n(x)/Q_n = \infty$. *W. J. LeVeque* (Ann Arbor, Mich.)

6357:

Šidlovskiĭ, A. B. Transcendentality and algebraic independence of the values of certain functions. Trudy Moskov. Mat. Obšč. **8** (1959), 283–320. (Russian)

Recently [Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 35–66; MR **21** #1295] the author published a detailed proof of his important theorem on the algebraic independence of the values of E-functions which satisfy a system of differential equations

$$(1) \qquad y_k' = Q_{k0}(z) + \sum_{l=1}^{m} Q_{kl}(z) y_l \quad (k = 1, 2, \cdots, m)$$

with rational functions $Q_{kl}(z)$ as coefficients. He now applies this theorem to prove a number of new results on the transcendency of the values of certain generalised hypergeometric functions. The simplest of his theorems is as follows. Let λ_0 be a non-negative integer; let $\lambda_1, \cdots, \lambda_m$ be non-integral rational numbers such that their differences $\lambda_{j_1} - \lambda_{j_2}$ $(1 \leq j_1 < j_2 \leq m)$ are not integers; let $\beta_1, \cdots, \beta_n$ be algebraic numbers linearly independent over the rational field; let $\alpha_1, \cdots, \alpha_n$ be distinct algebraic numbers not zero; and let

$$\varphi_\lambda(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\lambda+1)(\lambda+2)\cdots(\lambda+k)}.$$

Then the $(m+1)n$ numbers

$$\varphi_{\lambda_0}(\beta_i), \quad \varphi_{\lambda_j}(\alpha_i) \quad (j = 1, 2, \cdots, m; \quad i = 1, 2, \cdots, n)$$

are algebraically independent over the rational field. The proof consists of the following three steps. (i) The functions $\varphi_{\lambda_0}(\beta_i z)$ and $\varphi_{\lambda_j}(\alpha_i z)$ are E-functions. (ii) These functions satisfy a system of linear differential equations of the form (1). (iii) These functions are algebraically independent over the field of rational functions of z.

K. Mahler (Manchester)

6358:

Szüsz, P. Über die Approximation einer reellen Zahl durch Brüche. Acta Math. Acad. Sci. Hungar. **10** (1959), 69–75. (Russian summary, unbound insert)

Refining the results of an earlier paper [same Acta **6** (1955), 203–212; MR **17**, 133], the following theorems are proved. (I) If α is an arbitrary irrational number and $\varepsilon > 0$, there is an $n_0 = n_0(\alpha, \varepsilon)$ such that for each $n \geq n_0$ there is a natural number x for which the relations $\|\alpha x\| < (5^{1/2}x)^{-1}$ and $n \leq x < \varepsilon n^2$ hold. (II) Let $g(x) \to 0$ monotonically as $x \to \infty$, and let $f(x)$ be an increasing function. Then there is an irrational number α and an increasing function $f_1(x)$ with the following properties: (a) for infinitely many natural numbers x, $f_1(x) = f(x)$, and (b) for infinitely many positive integers N the inequality $\|\alpha x\| > f_1(x)/x$ holds for all integers x with $N \leq x < g(N)N^2$. *W. J. LeVeque* (Ann Arbor, Mich.)

6359:

Maass, Hans. Über die Verteilung der zweidimensionalen Untergitter in einem euklidischen Gitter. Math. Ann. **137** (1959), 319–327.

Let $\mathfrak{T}$ be an n-dimensional lattice in Euclidean m-space and denote by $\mathfrak{L}(\mathfrak{T})$ the subspace determined by $\mathfrak{T}$, by $\langle\mathfrak{T}\rangle$ the class of lattices similar to $\mathfrak{T}$ and by $|\mathfrak{T}|$ the n-dimensional lattice volume of $\mathfrak{T}$. Let $\mathfrak{R}_0$ be the space of two-dimensional subspaces and $\mathfrak{G}_0$ the space of similarity classes of two-dimensional lattices. Measures (both denoted by I) are introduced in a natural way into $\mathfrak{R}_0$ and $\mathfrak{G}_0$. Let $\mathfrak{R}$ and $\mathfrak{G}$ be measurable sets in $\mathfrak{R}_0$ and $\mathfrak{G}_0$ respectively and denote by $\alpha_q(\mathfrak{S}; \mathfrak{R}, \mathfrak{G})$ the number of two-dimensional sublattices $\mathfrak{T}$ of the m-dimensional lattice $\mathfrak{S}$ satisfying $\mathfrak{L}(\mathfrak{T}) \in \mathfrak{R}$, $\langle\mathfrak{T}\rangle \in \mathfrak{G}$ and $|\mathfrak{T}| \leq q$. The author proves the asymptotic result

$$\alpha_q(\mathfrak{S}; \mathfrak{R}, \mathfrak{G}) \sim \frac{I(\mathfrak{R})I(\mathfrak{G})}{I(\mathfrak{R}_0)I(\mathfrak{G}_0)} \frac{(2\pi q)^m}{24|\mathfrak{S}|^2 m\Gamma(m-1)}$$

for $q \to \infty$. For $\mathfrak{R} = \mathfrak{R}_0$ this result was established by W. Roelcke [same Ann. **131** (1956), 260–277; MR **18**, 273]. The present paper follows the same method of proof but utilizes in addition recent results of the author about zeta-functions [ibid. **134** (1957), 1–32; MR **19**, 838].

A. Dvoretzky (Jerusalem)

6360:

Barnes, E. S. The construction of perfect and extreme forms. II. Acta Arith. **5** (1959), 205–222.

This article continues the work of a previous one [Acta Arith. **5** (1959), 57–79; MR **20** #6998] and describes another general method of constructing perfect and extreme forms. A quadratic form $g(x_1, x_2, \cdots, x_n, x_{n+1})$ is called an extension of f if f can be gotten from g by replacing x_{n+1} by 0. The object is to find an extension of a perfect form which is itself perfect.

The author denotes by Π_x the set of points $\mathbf{x}$ which, with the metric defined by the positive definite form $f(\mathbf{x})$, are at least as near to the origin as to any other point of the integral lattice. A point of Π_x lying on n linearly independent faces is called a vertex $\mathbf{v}$ of Π_x. The principal theorem is: Suppose that $f(\mathbf{x})$ is perfect with minimum M, and that, for some integral $t \geq 1$, the point $t\boldsymbol{\lambda}$ is congruent to a vertex $\mathbf{v}$ of Π_x and satisfies

$$\mathscr{M}(t\boldsymbol{\lambda}) = \mathscr{M}(\mathbf{v}) < M.$$

Then the extension $g(\mathbf{x}, x_{n+1})$ defined by

$$g(\mathbf{x}, x_{n+1}) = f(x_1 + \lambda_1 x_{n+1}, \cdots, x_n + \lambda_n x_{n+1}) + cx_{n+1}^2$$

with $t^2 c = M - \mathscr{M}(t\boldsymbol{\lambda})$, has $M_s(g) = M$; and g is perfect if its minimum is M.

Various applications to particular forms are given.

B. W. Jones (Mayaguez)

6361:

Mahler, K. An arithmetic property of groups of linear transformations. Acta Arith. **5** (1959), 197–203.

The main result of this paper is the following theorem. Let

$$F = \left\{ s_k \middle| \; s_k(z) = \frac{\alpha_k z + \beta_k}{\gamma_k z + \delta_k}, \quad \alpha_k, \beta_k, \gamma_k, \delta_k \text{ real}, \; \alpha_k\delta_k - \beta_k\gamma_k = 1 \right\}$$

be a Fuchsian group with compact fundamental region. Let $f(u, v) = au^2 + 2buv + cv^2$ be a positive definite quadratic form. Then the values $f(\alpha_k, \gamma_k)$ lie everywhere dense in the positive real axis. This is in contrast to the situation for the modular group, where the values $f(\alpha_k, \gamma_k)$ are nowhere dense (and have a minimum

$$\mu_f = \min_{\alpha,\gamma \text{ integers}} f(\alpha, \gamma) \leqq (2/\sqrt{3})\sqrt{(ac - b^2)}).$$

The method of proof is as follows. If g is any form, represent g by the point z in the upper half plane such that $g(z, 1) = 0$. If f is represented by z, the F-equivalent form $f_k(u, v) = f(\alpha_k u + \beta_k v, \gamma_k u + \delta_k v)$ is represented by the point $z_k = x_k + iy_k = s_k^{-1}(z)$, where $y_k = \sqrt{(ac - b^2)}/f(\alpha_k, \gamma_k)$. The y_k are proved dense in the positive real numbers by showing that there is an image $s_k^{-1}(z)$ between any pair of horizontal lines. This follows from a theorem of G. A. Hedlund which states that if the fundamental region of F is compact, then the images $s_k(H)$ of any horocycle H are everywhere dense in the upper half plane.

The author conjectures analogous results when f is indefinite or when F is a Kleinian polyhedral group and f is Hermitian. The reviewer has proved these conjectures.

L. Greenberg (Providence, R.I.)

6362:

Tull, J. P. Dirichlet multiplication in lattice point problems. II. Pacific J. Math. **9** (1959), 609–615.

[For part I, see Duke Math. J. **26** (1959), 73–80; MR **21** #659.] The author generalizes his earlier result on the Stieltjes resultant $C(x)$ given in I. In the place of the functions $P_\mu(\log x)$ in asymptotic formulae for $A(x)$ and $B(x)$ [see the cited review], the author uses more general slowly oscillating functions in the sense of Karamata [Bull. Soc. Math. France **61** (1933), 55–62] with the canonical representation

$$L(x) = a \exp\left(\int_1^x t^{-1}\delta(t)dt\right), \quad a > 0, \quad \delta(x) \to 0, \quad x \to \infty,$$

δ is bounded and L-integrable on each bounded interval. Then the main terms in the Stieltjes resultant of $A(x)$ and $B(x)$, i.e., in $C(x) = \int_1^x A(x/u)dB(u)$, have the same form as the main terms in $A(x)$ and $B(x)$. Very precise O-remainder terms in asymptotic formulas for $C(x)$ are expressed by means of $x^\alpha L(x)$ and their integrals.

M. Tomić (Belgrade)

FIELDS

See also 6396.

6363:

Sancho de San Román, J. On subordinate valuations of a valuation of a field, in specializations of the latter. Rev. Mat. Hisp.-Amer. (4) **18** (1958), 244–256. (Spanish. French summary)

The author gives a condition for a subordinate valuation v' of a valuation v to be trivial and shows that if a valuation v' is given, then there exists v such that v' is subordinated to v.

E. Lluis (Mexico, D.F.)

6364:

Heerema, Nickolas. On ramified complete discrete valuation rings. Proc. Amer. Math. Soc. **10** (1959), 490–496.

We consider so-called p-adic valuation rings, that is, those complete discrete valuation rings which have characteristic 0 and whose residue class fields have characteristic p. It is well known that there exists a 1-1 correspondence (up to isomorphism) between unramified p-adic valuation rings R and fields F of characteristic p, by associating R with its residue class field $F = R/(p)$ [Witt, J. Reine Angew. Math. **176** (1937), 126–140; Teichmüller, ibid., 141–152; MacLane, Ann. of Math. (2) **40** (1939), 423–442]. This paper is concerned with some characterizations of ramified p-adic valuation rings in terms of unramified R based on the following main theorem: Let F be a field of characteristic p, R the corresponding unramified p-adic valuation ring and $R[[x]]$ the ring of power series over R; then p-adic valuation rings R_n with ramification index n and with residue class field F and only these are represented as residue class rings $R[[x]]/I$ modulo principal ideals I generated by elements of the form $p - ux^n$ with units u in $R[[x]]$. This, together with the fact that a perfect field has no nontrivial derivations, provides a simpler proof to the Teichmüller theorem that R is uniquely imbedded into R_n if F is perfect. Furthermore, it is also derived, for example, that if F is perfect and n is prime to p then every R_n is obtained by adjoining to R an nth root of an element of the form pa with unit a in R: $R_n = R[\sqrt[n]{(pa)}]$, and two such rings $R[\sqrt[n]{(pa)}]$ and $R[\sqrt[n]{(pb)}]$ are isomorphic if and only if there exist an automorphism π of R and an element z in R such that $a^\pi z^n \equiv b \pmod{p}$.

G. Azumaya (Sapporo)

6365:

Nikodým, Otton Martin. A study in algebraic extensions of linearly ordered non archimedean fields. J. Math. Pures Appl. (9) **38** (1959), 61–96.

Let F be a commutative field of characteristic zero. By the 'star-field' extension of F the author means the appropriate subset of mappings of the positive and negative integers into F and the appropriately defined multiplication and addition of these mappings to make the resulting system isomorphic to the field of formal power series in X over F with only finitely many terms of negative power. The author defines formally a differentiation in the star-field extension of F and a partial differentiation with respect to Y of polynomials in Y over the star-field. Using a similar formal definition of fractional power series of the Puiseux type with coefficients in F, the author proves that the result is isomorphic to the algebraic closure of the star-field extension, if F is algebraically closed. The detailed proof uses no topological apparatus beyond what can be set up by choosing a closed real subfield of F.

S. Gorn (Philadelphia, Pa.)

6366:

Kolchin, E. R. Rational approximation to solutions of algebraic differential equations. Proc. Amer. Math. Soc. **10** (1959), 238–244.

The author formulates and proves an interesting differential-algebraic generalization of Liouville's classical result about a limit to the accuracy with which algebraic numbers can be approximated by rational numbers and, thereby, opens the way for new types of investigations in differential algebra.

First, he generalizes the notion of the degree of an algebraic number as follows: If s is a natural number, then $\sum_{j=1}^{k}(j+s)e_j$ is called the s-denomination of the D-(differential-) monomial $\prod_{j=1}^{k}(D^jy)^{e_j}$ in a D-indeterminate y. The s-denomination of a non-zero D-polynomial is then defined to be the maximum of the s-denominations of the D-monomials of which it is a linear combination with non-zero coefficients. Now an element u which is D-algebraic over a commutative D-ring Z is said to have s-denomination d over Z if d is the smallest natural number such that there exists a non-zero D-polynomial P in $Z\{y\}$ of s-denomination d vanishing at u.

Theorem: Let F be a field with a derivation D and a non-trivial valuation $a\to|a|$ for which there exist elements α, β of the value group such that $\alpha|a|\leqq|Da|\leqq\beta|a|$ holds for all $a\in F$ with $|a|<1$. Let Z be a non-zero D-subring of F with $|a|\geqq 1$ for all its non-zero elements a. Then for any $u\in F$ which is D-algebraic over Z and any natural number s there exists an element γ of the value group such that $|u-a/b^s|\geqq\gamma/|b|^d$ for all a, $b\in Z$ with $b\neq 0$ and $u\neq a/b^s$. The main part of the proof consists in showing that there exists an element γ_0 of the value group such that $|v-u|\geqq\gamma_0$ for every $v(\neq u)\in F$ which is a zero of a polynomial P of minimal s-denomination vanishing at u. For this purpose some properties of Wronskian determinants are established and applied. From the theorem it follows that a power series $\sum_{k=0}^{\infty}c_kx^{s_k}$ with non-zero coefficients c_k in a field K of characteristic 0, and with strictly increasing integral exponents $s_k>0$ such that the sequence of ratios s_{k+1}/s_k is unbounded, is D-transcendental over $K(x)$.

A. Jaeger (Cincinnati, Ohio)

6367:

Rosenfeld, Azriel. Specializations in differential algebra. Trans. Amer. Math. Soc. **90** (1959), 394–407.

The paper shows how a particular type of chain of partial D-(differential-) polynomials in the sense of Ritt [*Differential algebra*, Amer. Math. Soc., New York, 1950; MR **12**, 7; p. 164] can be employed skillfully to reduce theorems on D-polynomial rings to analogous theorems for suitable polynomial rings. Chapter I defines ranking of the set of all (including improper and higher) partial derivatives of D-indeterminates by the properties of a complete system of marks of Riquier [Ritt, loc. cit., p. 151]. Leaders, initials, separants, the reduction process, etc., are then considered in terms of some ranking. The author makes the good suggestion to use the word "autoreduced set" for the concept called "chain" by Ritt and introduces the notion of coherence for such sets. A coherent autoreduced set is characterized as a chain A such that any D-polynomial in $[A]$: $(IS)^\infty$ which is free of proper derivatives of the leaders of A must actually be in (A): $(IS)^\infty$, where $(IS)^\infty$ stands for some sufficiently high power of the product $\prod I_iS_i$ of the initials and separants of A. Chapter II contains two theorems on extensions of specializations over D-fields: There exists a D-polynomial such that any specialization, not annihilating it, can be extended to a specialization not annihilating a given D-polynomial. Under certain "properness" conditions an intermediate specialization of parametric D-indeterminates can be extended to an intermediate specialization. Chapter III brings applications to chains of prime D-ideals, a theorem on the dimensions of certain prime D-ideal components of the perfect D-ideal generated by a coherent autoreduced set, and furnishes the first steps toward the development of a theory of the dimensions of the components of the intersection of D-varieties. *A. Jaeger* (Cincinnati, Ohio)

ALGEBRAIC GEOMETRY

6368:

Samuel, Pierre. Un exemple de variété affine normale. Soc. Parana. Mat. Anuário (2) **1** (1958), 4–5. (Portuguese summary)

If n is an integer prime to the characteristic of the field K and if the polynomial $F(X_1, X_2, \cdots, X_s)$ has no irreducible factor of multiplicity greater than n, then the hypersurface defined by the equation $Y^n-F(X_1, X_2, \cdots, X_s)=0$ is shown to be normal in the $(s+1)$-dimensional affine space over K, provided it is irreducible.

H. T. Muhly (Iowa City, Iowa)

6369:

Godeaux, Lucien. Note sur les surfaces de genres zéro possédant un réseau irréductible de courbes bicanoniques. I, II. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 52–68.

Principal result: If a surface has genera $p_a=p_g=0$, $p^{(1)}=P_2=3$ and has an irreducible pencil of (in general) irreducible bicanonical curves, then it contains a curve Γ such that 2Γ is bicanonical, and, if Γ' is adjoint to Γ, then $\Gamma+\Gamma'$ is tricanonical and $2\Gamma'$ is 4-canonical.

Campedelli constructed a surface with these properties: "Sui piani doppi con curva di diramazione del decimo ordine" [Atti Accad. Naz. Lincei Rend. (6) **15** (1932), 358–362]. The author shows that the branch curve named in the title breaks up into four curves, a quartic and three conics, which touch each other by sets of three at six distinct points. He uses as a projective model the tricanonical surface, F, in S_6, of order 18 and hyperplane section genus 13. F always lies on two linearly independent quadric primals and may lie on 3 or 4 such primals. In the case when F lies on 2 or 3 quadrics only, it has on it six distinct curves $\Gamma_1, \cdots, \Gamma_6$ such that $\Gamma_1+\Gamma_2\equiv\Gamma_3+\Gamma_4\equiv\Gamma_5+\Gamma_6$ are bicanonical, $\Gamma_1+\Gamma_3+\Gamma_5\equiv\Gamma_2+\Gamma_4+\Gamma_6$ are tricanonical and $\Gamma_1+\cdots+\Gamma_6$ is 6-canonical.

T. G. Room (Sydney)

6370:

Shimura, Goro. On the theory of automorphic functions. Ann. of Math. (2) **70** (1959), 101–144.

For a given Z-algebra $\mathfrak{r}$ with finite basis, define a polarized abelian variety of type $\mathfrak{r}$ to be an abelian variety A with a given polarization (meaning, roughly, a class of projective embeddings) and a given isomorphism from $\mathfrak{r}$ into the endomorphism ring of A. For such data a field of moduli is defined; this is, roughly, the least field over which an isomorphic polarized variety of type $\mathfrak{r}$ can be

defined. If now we are given an analytic family of polarized abelian varieties of type $\mathfrak{r}$, parametrized by a connected open subset $\mathfrak{Z}$ of C^m (that is, for each $z \in \mathfrak{Z}$ we are given a Riemann matrix of specified type, etc.), then there exist meromorphic functions $f_1(z), \cdots, f_n(z)$ on $\mathfrak{Z}$ and an analytic subset $\mathfrak{G}$ of $\mathfrak{Z}$, of codimension one and containing the polar locus of each f_i, such that $Q(f_1(z), \cdots, f_n(z))$ is the field of moduli of the polarized abelian variety of type $\mathfrak{r}$ corresponding to any point $z \in \mathfrak{Z} - \mathfrak{G}$; in addition, k being a countable subfield of C, a condition is given for $k(f_1, \cdots, f_n)$ to be a regular extension of k, linearly disjoint from C over k. The last part of the paper considers polarized abelian varieties of dimension 2 with endomorphism rings isomorphic to an order of an indefinite quaternion algebra, i.e., a central simple Q-algebra of degree 2 possessing a faithful representation by real 2×2 matrices; analytic families of such polarized abelian varieties are constructed, parametrized by the upper half plane of C, for which the moduli are precisely the automorphic functions on the upper half plane corresponding to a certain transformation group, and the field of moduli is shown to be defined over Q.

M. Rosenlicht (Berkeley, Calif.)

6371:

★**Shimura, Goro. Fonctions automorphes et variétés abéliennes.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 167, 9 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

This can be considered as an exposé of the special case of the paper reviewed above, in which one considers the family of polarized abelian varieties given by the Siegel variety; the field of moduli here is precisely the field of modular functions of Siegel. There is also a brief discussion of the modular functions obtained from the division points of a certain order on a variable abelian variety.

M. Rosenlicht (Berkeley, Calif.)

LINEAR ALGEBRA

See also 6333, 6336, 6560, 6688.

6372a:

★**Gantmacher, F. R. Matrizenrechnung. II. Spezielle Fragen und Anwendungen.** Hochschulbücher für Mathematik, Bd. 37. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. viii + 244 pp. DM 26.00.

6372b:

★**Gantmacher, F. R. Applications of the theory of matrices.** Translated by J. L. Brenner, with the assistance of D. W. Bushaw and S. Evanusa. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1959. ix + 317 pp. $9.00.

6372c:

★**Gantmacher, F. R. The theory of matrices. Vols. 1, 2.** Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959. Vol. 1, x + 374 pp. $6.00; Vol. 2, ix + 276 pp. $6.00.

These books are translations of *Teoriya matric* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 438]. The German translation of part I is listed as MR **20** #3884. The translation by J. L. Brenner is of part II (Chapters XI to XV) which contains the most original material, to which the translator has added references, an index and four appendices; he has also corrected misprints, simplified a few proofs, and made the book a self-contained whole. The translation by K. A. Hirsch is of the complete Russian text with new versions of several paragraphs communicated by the author.

The work is an outstanding contribution to matrix theory and contains much material not to be found in any other text.

6373:

★**Parker, William Vann; and Eaves, James Clifton. Matrices.** The Ronald Press Co., New York, 1960. viii + 195 pp. $7.50.

Introductory textbook. Matrices are introduced in terms of forms; vector spaces do not appear until the middle of the book, the notion of duality not at all, and in general the development is via coordinates (i.e., not coordinate-free). Exercises are rather extensive, with some emphasis on computation, although "extreme care was used ... to avoid complete entanglement in manipulation". Contents in order: Brief review of elementary concepts; algebra of matrices; transformations of matrices; bilinear forms and rank; systems of equations; congruence of square matrices; quadratic forms; vector space over a field; determinants; characteristic polynomials, elementary divisors, etc.; special matrices.

6374:

Cardoso, Jayme Machado. Note on finite vector spaces. Soc. Parana. Mat. Anuário (2) **1** (1958), 2–3. (Portuguese. English summary)

Some elementary results on the theory of vector spaces constructed on finite groups.

R. G. Stanton (Waterloo, Ont.)

6375:

★**Asplund, Edgar. Iteration of matrices over a distributive lattice.** Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 19–27. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Matrices whose elements belong to some distributive lattice are considered. The product of two matrices $G = \{\gamma_{ik}\}$ and $H = \{\eta_{ik}\}$ is defined as the matrix $GH = \{\bigvee_j (\gamma_{ij} \cap \eta_{jk})\}$. The functions

$$m(G) = \text{Min}\,\{\mu \mid G^\mu = G^{\mu+\nu} \text{ for some } \nu\},$$
$$p(G) = \text{Min}\,\{\pi \mid G^\nu = G^{\nu+\pi} \text{ for some } \nu\}$$

(μ, ν, π are natural numbers) are investigated. Theorems: The maxima of the functions $m(G)$ and $p(G)$ defined on the class of $n \times n$ matrices over a general distributive lattice are $(n-1)^2+1$ and l.c.m. $\{1, 2, \cdots, n\}$ (least common multiple) respectively. The author first establishes the corresponding theorems in the case of the two-element lattice **2**, and then uses the representation theory for the general case.

M. Kolibiar (Bratislava)

6376:

Bachman, G.; and Hellman, M. J. On the parametrization of the proper orthogonal group. Arch. Math. **10** (1959), 93–100.

The authors construct polynomial mappings from subsets of the space of n by n skew symmetric matrices to the space of n by n orthogonal matrices. These mappings are effectively one-to-one and in their totality are onto.

H. S. A. Potter (Aberdeen)

6377:

Brenner, J. L.; and Cooke, K. L. Hadamard matrices. Rev. Ci. Lima **59** (1957), 5–13.

If A is a non-singular matrix with complex elements, there exists a permutation matrix P and an upper triangular matrix T such that $C=APT$ has dominant principal diagonal, i.e., $|c_{ii}|>\sum_{j\neq i}|c_{ij}|$ for all i.

H. S. A. Potter (Aberdeen)

6378:

Gun, G. The argument of the maximum absolute value of characteristic roots of a matrix. Acta Math. Sinica **9** (1959), 170–173. (Chinese. English summary)

For a square matrix A with real or complex elements, let

$$\frac{1}{2}(A^k+A^{*k})=(b_{ij}^{(k)}),\quad \frac{1}{2i}(A^k-A^{*k})=(c_{ij}^{(k)}),$$

$$b_k=\sum_i\sum_j|b_{ij}^{(k)}|^2,\quad c_k=\sum_i\sum_j|c_{ij}^{(k)}|^2.$$

If ω is an eigenvalue of A of maximum modulus, the author's theorem asserts that $\arg\omega$ is one of the values $\pm\psi$ and $\pi\pm\psi$, where $0\leq\psi\leq\frac{1}{2}\pi$, $\psi=\lim_{k\to\infty}\tan^{-1}u_k$ and

$$u_k=\frac{|(b_{k+1}c_k)^{1/2}-(b_kc_{k+1})^{1/2}|}{(b_{k+1}b_k)^{1/2}+(c_{k+1}c_k)^{1/2}}\quad\text{or}\quad\frac{(b_{k+1}c_k)^{1/2}+(b_kc_{k+1})^{1/2}}{|(b_{k+1}b_k)^{1/2}-(c_{k+1}c_k)^{1/2}|}.$$

This theorem is false. First, it may happen that the sequence $\{\tan^{-1}u_k\}$ fails to converge no matter how one of the two choices for each u_k is made. Furthermore ψ may be distinct from every cluster point of the sequence $\{\tan^{-1}u_k\}$ for any combination of choices of u_k. Consider, for example, the eigenvalue $\omega=e^{2\pi i/3}$ of

$$A=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

Ky Fan (Notre Dame, Ind.)

6379:

Gun, G. Limits for the characteristic roots of a matrix. II. Acta Math. Sinica **9** (1959), 174–180. (Chinese. English summary)

This is a sequel to the author's earlier paper [Advancement in Math. **4** (1958), 450–456; MR **20** #3893]. Let $A=(a_{ij})$ be a real or complex n-square matrix. Let $a=\max_{i\neq j}|a_{ij}|$,

$$e=\frac{1}{2n^2}\sum_i\sum_j(|a_{ij}|-|a_{ji}|)^2,\quad r_i=\sum|a_{ij}|,$$

$s=\max_i r_i-\min_i r_i$. For each positive integer k, let

$$a_k=\operatorname{tr}A^kA^{*k},\quad b_k=\operatorname{tr}\left(\frac{A^k+A^{*k}}{2}\right)^2,$$

$$c_k=\operatorname{tr}\left(\frac{A^k-A^{*k}}{2i}\right)^2.$$

It is shown that each eigenvalue of A has an absolute value at most equal to $[a_1-a^4e/(s+a)^4]^{1/2}$. For each k, a closed curve Γ_k in the complex plane is defined in terms of a_k, b_k, c_k such that all eigenvalues of A lie inside or on Γ_k. There is also a result comparing the difference $\sum_{i=1}^n|\lambda_i|^{2k}-\max(b_k,c_k)^n$ with $\min(b_k,c_k)^n$, where λ_i are the eigenvalues of A.

Ky Fan (Notre Dame, Ind.)

6380:

Householder, Alston S. Minimal matrix norms. Monatsh. Math. **63** (1959), 344–350.

If $\rho(A)$ denotes the maximum of the absolute values of the eigenvalues of the matrix A and $\|A\|$ is a norm for A, then $\rho(A)\leq\|A\|$. If $\|A\|=\rho(A)$ the norm is called minimal for A. Minimal norms are constructed for certain classes of matrices.

H. S. A. Potter (Aberdeen)

6381:

Picone, Mauro. Sulle maggioranti i numeri caratteristici di una matrice quadrata. Boll. Un. Mat. Ital. (3) **13** (1958), 335–340. (English summary)

For any square matrix a of order n, a real function $\phi(a)$ is said to majorize the characteristic roots μ of a if for every μ the relation $|\mu|\leq\phi(a)$ is satisfied. If ϕ_1 and ϕ_2 both majorize the characteristic roots of a, and if $\phi_1(a)\leq\phi_2(a)$ for every a, then ϕ_1 is said to be stronger than ϕ_2. Several such functions are considered and compared [cf. also #6688]. It may be remarked that among all matrix norms consistent (in the sense of Faddeeva) with a given vector norm, the one that is subordinate is stronger than all others.

A. S. Householder (Oak Ridge, Tenn.)

6382:

Farahat, H. K.; and Ledermann, W. Matrices with prescribed characteristic polynomials. Proc. Edinburgh Math. Soc. **11** (1958/59), 143–146.

Let n be an integer ≥ 2; Φ an arbitrary field; and $f(x)$ a given monic polynomial of degree n in $\Phi[x]$. The two main results established in the present paper are as follows. (i) Let $a_1,\cdots,a_{n-1}\in\Phi$. Then there exists an $n\times n$ matrix over Φ with characteristic polynomial $f(x)$ and having $a_1,\cdots,a_{n-1}$ on its main diagonal. For the special case when Φ is the field of real or of complex numbers, this had been noted by the reviewer [J. London Math. Soc. **33** (1958), 14–21; MR **19**, 1034]. (ii) Let B be an $(n-1)\times(n-1)$ non-derogatory matrix over Φ. Then there exists an $n\times n$ matrix over Φ with characteristic polynomial $f(x)$ and having B in its left-hand corner.

The proofs of these two results are independent; but it is also shown, in conclusion, that (i) is a simple consequence of (ii).

L. Mirsky (Sheffield)

6383:

Farnell, A. B. A special Vandermondian determinant. Amer. Math. Monthly **66** (1959), 564–569.

Let $a_{ij}=\cos^j(\pi i/m)$ for $i=0,1,\cdots,m$; $j=1,\cdots,m$, and let $a_{i0}=1$ for $i=0,1,\cdots,m$; then $\det(a_{ij})=m^{m+1}/2^{m^2-3}$.

H. S. A. Potter (Aberdeen)

6384:

★Bourbaki, N. Éléments de mathématique. Première

partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 9: Formes sesquilinéaires et formes quadratiques. Actualités Sci. Ind. no. 1272. Hermann, Paris, 1959. 211 pp. (1 insert)

This stimulating book is a reflection of the new life in an old subject. Besides treating the classical theory of quadratic and Hermitian forms with unusual generality, it deals with other less prominent topics such as the Clifford algebra and spinor norm. There are numerous exercises, some illustrative, others considerable extensions of the text (e.g. the 'fundamental theorem of projective geometry', Frobenius and symmetric algebras). Text and exercises together give quite a comprehensive account of the subject. The treatment leans to some extent on the earlier chapters of Bourbaki's *Algèbre*, chiefly the basic ones on linear and multilinear algebra. The inclusion of 'plane trigonometry' in the final section of this ninth chapter is interesting. The book concludes with an excellent historical note.

Outline of Contents. § 1. Generalities on sesquilinear mappings and forms. § 2. Discriminants, with application to algebras. § 3. Generalities on Hermitian forms (over a ring with involutory antiautomorphism) and quadratic forms (over a commutative ring). § 4. Witt's theorem for quadratic and Hermitian forms over fields. {Remark: the form of proof adopted rather obscures the fundamental rôle of the trace condition (T).} § 5. Alternate forms. Reduction to canonical form over a commutative principal ideal ring. Pfaffian, symplectic group, over a commutative ring. § 6. Hermitian forms. Orthogonal and orthonormal bases. Symmetries and similitudes. § 7. Hermitian forms over 'real', 'complex' and 'quaternion' fields. Law of inertia. Normal matrices. § 8. Ring of types of a quadratic form [cf. E. Witt, J. Reine Angew. Math. **176** (1937), 31–44]. § 9. Clifford algebra and group, spinor norm. § 10. Angles, sectors, plane trigonometry. (In Euclidean plane geometry, one can derive cosine and sine from the matrix $\cos\theta I+\sin\theta J$ $(J^2=-I)$ of a rotation; here, a general cosine and sine are derived from a more general 2-dimensional orthogonal group.) *G. E. Wall* (Sydney)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 6364, 6390, 6391, 6396.

6385:

Tachikawa, Hiroyuki. On rings for which every indecomposable right module has a unique maximal submodule. Math. Z. **71** (1959), 200–222.

A ring A ($\ni 1$) with right minimum condition is said to be of strong right cyclic representation type when every right A-module is a homomorphic image of a right ideal generated by a primitive idempotent. Firstly, on considering the endomorphism ring B of a finitely generated faithful injective right A-module M such that every simple right A-module is isomorphic to a submodule of M, and on using the duality in Tachikawa [Math. Z. **68** (1958), 479–487; MR **20** #895], the problem of characterizing such a ring A is reduced to that of a ring B with minimum condition such that every directly indecomposable right B-module has a unique minimal submodule. Then the last problem is solved, after a complicated analysis, by certain four conditions; one is that every left ideal of B generated by a primitive idempotent should have a unique composition series while the other three are concerned with homomorphic images and composition factors of such left ideals. The result contains as a special case the reviewer's [Proc. Imp. Acad. Tokyo **16** (1940), 285–289; MR **2**, 245] result on rings having the property quoted at the beginning of the review and its right-left counterpart.

T. Nakayama (Nagoya)

HOMOLOGICAL ALGEBRA

6386:

Sridharan, R. A generalization of the almost zero theory. Nagoya Math. J. **14** (1959), 85–93.

B. Eckmann introduced in Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 275–281 [MR **9**, 244], the almost zero theory for the cohomology of groups. In this note a generalization of this theory to the theory of algebras is given. Let Λ be an algebra over K (a commutative ring) and C a right Λ-module. The author defines $\bar{H}^n(\Lambda, C)$ to be $H^n(\Lambda, \Lambda\otimes_K C)$, where $\Lambda\otimes_K C$ is considered as a two-sided Λ-module by means of the left operation of Λ on Λ and the right operation of Λ on C. Following the usual homological practice, the author defines $\dim^{\sim}\Lambda$ to be the largest n (if it exists) such that $\bar{H}^n(\Lambda, C)\neq 0$ for some C. If no such integer exists, he defines $\dim^{\sim}\Lambda=\infty$. It is then shown that if the $\dim\Lambda$ is finite, then $\dim^{\sim}\Lambda=\dim\Lambda$. The rest of the paper is devoted primarily to showing that for a certain class of algebras the groups $\bar{H}^n(\Lambda, C)$ can be described in analogous way to Eckmann's original definition of the almost zero theory of groups.

M. Auslander (Waltham, Mass.)

GROUPS AND GENERALIZATIONS

See also 6376.

6387:

Novikow, P. S. Über einige algorithmische Probleme der Gruppentheorie. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 88–92.

Non-technical survey article.

6388:

Rosenbaum, R. A. Some simple examples of groups. Amer. Math. Monthly **66** (1959), 902–905.

There are given several examples of groups defined upon certain subsets of the complex plane with operations most of which are of the form

$$u * v = (Luv+M(u+v)+N)/(Puv+Q(u+v)+R).$$

Their characteristic properties and isomorphisms between them and the addition or multiplication groups of certain sets of complex numbers are investigated. [For related considerations see, e.g., W. Alt, Deutsche Math. **5** (1940), 1–12; MR **1**, 297; A. Kuwagaki, Mem. Coll. Sci. Univ. Kyoto Ser. A **26** (1951), 139–144; MR **13**, 466.] The present paper is written in an agreeable conversational style almost entirely without proofs of the easily verifiable assertions.

J. Aczél (Debrecen)

6389:

Harary, Frank. Exponentiation of permutation groups. Amer. Math. Monthly **66** (1959), 572–575.

This paper deals with some permutational equivalences between certain combinations of "direct sum" $\mathfrak{A}+\mathfrak{B}$, "cartesian product" $\mathfrak{A}\times\mathfrak{B}$, "composition" $\mathfrak{A}[\mathfrak{B}]$ and "exponentiation" $\mathfrak{B}^{\mathfrak{A}}$ of permutation groups. Two permutation groups $\mathfrak{A}$ and $\mathfrak{B}$ with disjoint object sets X and Y are permutationally equivalent: $\mathfrak{A}=\mathfrak{B}$, if $\mathfrak{A}$ and $\mathfrak{B}$ are abstractly isomorphic and there is a one to one correspondence $f: X\leftrightarrow Y$ such that if θ is the abstract isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then for all $x\in X$, $\alpha\in\mathfrak{A}$, $f(\alpha x)=(\theta\alpha)f(x)$. Besides stating some simple identities involving these combinations of permutation groups, the author proves the following. (1) Among the operations of direct sum, cartesian product, and composition, the only distributive law which holds is $(\mathfrak{A}+\mathfrak{B})[\mathfrak{C}]=\mathfrak{A}[\mathfrak{C}]+\mathfrak{B}[\mathfrak{C}]$. (2) $\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}}=\mathfrak{A}^{\mathfrak{B}}\times\mathfrak{A}^{\mathfrak{C}}$, $(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}}=\mathfrak{A}^{\mathfrak{C}[\mathfrak{B}]}$.

The operation of exponentiation of two permutation groups has applications in the enumeration of types of Boolean functions and a certain class of bicolored graphs.

S. Lin (Athens, Ohio)

6390:

Beaumont, R. A.; and Pierce, R. S. Partly transitive modules and modules with proper isomorphic submodules. Trans. Amer. Math. Soc. **91** (1959), 209–219.

An R-module M over a principal ideal domain R is called an I-module if M contains a proper isomorphic submodule. The paper deals with the problem of determining all I-modules. The case of divisible and that of reduced modules can be considered separately. The divisible R-module M is an I-module if and only if there exists a primary component P_j of the torsion submodule T of M such that $M/\sum_{i\neq j}P_i$ has infinite rank. A torsion-free R-module which is not divisible is an I-module. An R-module M of finite rank is an I-module if and only if M/T is not divisible, where T is the torsion submodule of M. Let $\aleph$ be a cardinal number. $|M|=\aleph$ denotes that the R-module M has a generating set with $\aleph$ elements and $\aleph$ is the least cardinal with this property. $\|S\|$ is the cardinal number of the set S. The p-primary reduced R-module M is an I-module if either of the following conditions holds: (i) $|M|>\|R\|^{\aleph_0}$, (ii) $|M|=\aleph_0$. This last result is based on the concept of partly transitive module, for which see the next review.

A. Kertész (Debrecen)

6391:

Beaumont, R. A.; and Pierce, R. S. Partly invariant submodules of a torsion module. Trans. Amer. Math. Soc. **91** (1959), 220–230.

A submodule of a module M is called partly invariant in M if it is mapped into itself by every isomorphism of M into M. Let M be a torsion module over a principal ideal ring R. The problem of determining all partly invariant submodules of M is reduced to the primary case. Suppose M to be a p-primary R-module of length λ and let μ be the least ordinal such that $\sum_{\mu\leq\alpha<\lambda}f(\alpha)<\aleph_0$ if $f(\lambda)<\aleph_0$, and put $\mu=\infty$ otherwise. [For the standard terminology and notations see Kaplansky's book, *Infinite abelian groups*, Univ. of Michigan Press, Ann Arbor, 1954; MR **16**, 444.] $H_\mu=M_\mu$ if $\mu\leq\lambda$ and $=0$ if $\mu=\infty$, is called the top submodule of M. Let $U_1=(\alpha_0, \alpha_1, \alpha_2, \cdots)$, $U_2=(\beta_0, \beta_1, \beta_2, \cdots)$ be infinite U-sequences for M; then write $U_1\dashv U_2$ if $\alpha_i\leq\beta_i$ for all i and $\alpha_i=\beta_i$ if $\beta_i\geq\mu$. M is called partly transitive if, whenever $x, y\in M$ and $U(x)\dashv U(y)$, there exists an isomorphism φ such that $\varphi(x)=y$. The paper gives a profound investigation of the partly invariant submodules of the partly transitive p-primary R-module M. The main result can be summarized as follows. Let $(\alpha_0, \alpha_1, \cdots, \alpha_{N-1})$ be a finite U-sequence of ordinals less than μ and T be a characteristic submodule of the top submodule of M. Then $S=[x\in M\,|\,h(p^ix)\geq\alpha_i$ for $0\leq i<N$ and $p^Nx\in T]$ is a partly invariant submodule of M; and conversely, every partly invariant submodule of M not of this form must be fully invariant.

A. Kertész (Debrecen)

6392:

Kargapolov, M. I. Some problems in the theory of nilpotent and solvable groups. Dokl. Akad. Nauk SSSR **127** (1959), 1164–1166. (Russian)

From the results which the author announces, without proofs, in this note we select one that is positive and two that are negative; we postpone a detailed account until the proofs are available in full. (1) Every locally soluble group of finite rank contains a subgroup of finite index whose derived group has a central series. (2) Let the index α range over all finite and countable ordinals. For every α we choose a cyclic group P_α of order p, a prime, and we form the "inverse" wreath product G of all the P_α. This is defined inductively by $G_\alpha=P_\alpha\wr G_{\alpha-1}$ with the obvious provision for limit ordinals. This group G turns out to be a locally finite p-group which has no ascending normal series with abelian factors. It follows that a locally nilpotent group need not be an SN^*-group and this settles a long outstanding open question. (3) P. Hall's theorems for finite soluble groups (on the existence of complete Sylow bases and the complementation of Sylow subgroups) have been extended by various authors to certain classes of periodic locally soluble groups. But the present author constructs an example of a periodic two-step soluble group which has no complete Sylow basis. This example, too complicated to be reproduced here, also shows that Schur's Theorem does not extend to all locally finite groups and that the Sylow p-subgroups of a periodic soluble group need not be complemented.

K. A. Hirsch (London)

6393:

Gaschütz, Wolfgang. Die Eulersche Funktion endlicher auflösbarer Gruppen. Illinois J. Math. **3** (1959), 469–476.

Let $\mathfrak{G}$ be a finite group, and define $\phi_{\mathfrak{G}}(s)$ to be the number of ordered sets of s elements of $\mathfrak{G}$ (not necessarily distinct) which generate $\mathfrak{G}$. The function $\phi_{\mathfrak{G}}(s)$ is one of the Eulerian functions introduced by P. Hall [Quart. J. Math. Oxford Ser. (2) **7** (1936), 134–151] and calculated by him for certain simple groups $\mathfrak{G}$. The present paper gives, for any soluble $\mathfrak{G}$, an explicit formula for $\phi_{\mathfrak{G}}(s)$ in terms of invariants arising from a principal series of $\mathfrak{G}$.

Let $H: \mathfrak{G}=\mathfrak{R}_1\supset\mathfrak{R}_2\supset\cdots\supset\mathfrak{R}_r\supset 1$ be a principal series of $\mathfrak{G}$, and let $\mathfrak{E}_1, \mathfrak{E}_2, \cdots, \mathfrak{E}_h$ be the distinct isomorphism types of the factors of H when considered as $\mathfrak{G}$-groups. Notation: $p_i^{\gamma_i}$ is the order of $\mathfrak{E}_i$ (p_i prime); $\zeta_i=0$ or 1 according as $\mathfrak{G}$ acts trivially on $\mathfrak{E}_i$ or not; $p_i^{\sigma_i}$ is the number of $\mathfrak{G}$-endomorphisms of $\mathfrak{E}_i$; finally, of the factors of H of type $\mathfrak{E}_i$, α_i are complemented in $\mathfrak{G}$ and β_i are not (here $\mathfrak{R}_j/\mathfrak{R}_{j+1}$ is complemented in $\mathfrak{G}$ if there is a subgroup

$\mathfrak{R}$ of $\mathfrak{G}$ with $\mathfrak{R}_j\mathfrak{R}=\mathfrak{G}$ and $\mathfrak{R}_j\cap\mathfrak{R}=\mathfrak{R}_{j+1}$). Then

$$\phi_{\mathfrak{G}}(s)=\prod_{i=1}^{k}[p_i^{\lambda_i s}p_i^{s}(p_i^{\lambda_i s}-p_i^{\lambda_i c_i})(p_i^{\lambda_i s}-p_i^{\lambda_i c_i+\varkappa_i})\cdots \times(p_i^{\lambda_i s}-p_i^{\lambda_i c_i+(a_i-1)\varkappa_i})].$$

Further, two expressions of the form

$$t^s(p_1^{c_1 s}-d_1)(p_2^{c_2 s}-d_2)\cdots(p_m^{c_m s}-d_m),$$

where the p_j are primes, not necessarily distinct, take the same values for all positive integers s if, and only if, they are formally the same apart from the order of factors. This gives a criterion for deciding whether two soluble groups have the same Eulerian function $\phi_{\mathfrak{G}}(s)$. In particular, if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are soluble, and if $\phi_{\mathfrak{G}_1}(s)=\phi_{\mathfrak{G}_2}(s)$ for all s, then $\mathfrak{G}_1$ and $\mathfrak{G}_2$ have the same order and the same number of maximal subgroups. That they need not be isomorphic is shown by an example. *P. J. Higgins* (London)

6394:

Gol'berg, P. A. Hall Θ-bases for finite groups. Dokl. Akad. Nauk SSSR **127** (1959), 494–496. (Russian)

Let θ be a finite system of disjoint subsets of the set of prime numbers, $\theta=\{\Pi_1,\Pi_2,\cdots,\Pi_k\}$. A set of subgroups $H_1,H_2,\cdots,H_k$ of a group G shall be called a Hall θ-basis if the H_i are (i) Hall Π_i-subgroups of G (i.e., the prime divisors of their orders do belong and those of their indices do not belong to Π_i) and (ii) elementwise permutable. The following theorem is announced, without proof. Let G be a finite Π-soluble group, where Π is a finite set of primes, $\overline{\Pi}$ the set of prime divisors of the order of G that do not belong to Π and θ as above, but subject to the restriction that either all the intersections $\overline{\Pi}\cap\Pi_i$ are empty or θ contains a Π_s such that $\overline{\Pi}\subseteq\Pi_s$. Then G contains at least one Hall θ-basis and any two Hall θ-bases are conjugate. Also, every Hall θ-basis of a subgroup of G is obtained by intersecting the subgroup with the terms of a Hall θ-basis of G. Special cases: If G is soluble and θ the set of prime divisors of G, then the above theorem yields Hall's theorems on Sylow bases of soluble groups. If G is a Π-soluble group, $\Pi_1\subseteq\Pi$ and Π_2 the complement of Π_1 in all the primes, then with $\theta=\{\Pi_1,\Pi_2\}$ a theorem of Cunihin [Dokl. Akad. Nauk SSSR **73** (1950), 29–32; MR **12**, 156] arises. *K. A. Hirsch* (London)

6395:

Wielandt, Helmut. Zum Satz von Sylow. II. Math. Z. **71** (1959), 461–462.

G is a finite group, Π a set of prime numbers. A Π-Hall group of G is a subgroup G_Π of G whose order is divisible only by prime numbers in Π and whose index is not divisible by any prime in Π. The Π-Sylow theorem holds in G if G has a Π-Hall group containing a conjugate for every Π-subgroup of G. In the preceding paper with the same title [Math. Z. **60** (1954), 407–408; MR **16**, 331] the author has proved that the Π-Sylow theorem holds if G contains a nilpotent Π-Hall group. This is now sharpened to the following theorem: Let Π be the union of two disjoint sets σ,τ of prime numbers. Suppose that G contains a nilpotent σ-Hall group G_σ and a τ-Hall group G_τ which are elementwise permutable. Suppose further that the τ-Sylow theorem holds in G. Then the Π-Sylow theorem holds in G. The case previously treated is $\tau=\emptyset$. *K. A. Hirsch* (London)

6396:

Ribenboim, Paulo. Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation. Summa Brasil. Math. **4** (1958), 1–64.

Soient Γ un groupe abélien totalement ordonné et P un ensemble totalement ordonné isomorphe à l'ensemble de tous les sous-groupes isolés principaux (c'est-à-dire engendrés par un élément) de Γ ordonné par la relation inverse de l'inclusion. Soit Δ_σ le sous-groupe isolé engendré par $a\in\Gamma$ $(\sigma\in P)$. Il admet un prédécesseur immédiat (pour l'inclusion) Δ_σ^*. On note R_σ le groupe archimédien $\Delta_\sigma/\Delta_\sigma^*$. Le couple $(P,(R_\sigma)_{\sigma\in P})$ est dit squelette de Γ.

Γ est dit maximal s'il ne peut être agrandi sans que son squelette le soit. L'auteur montre, qu'étant donné un couple $(P,(R_\sigma)_{\sigma\in P})$, où P est un ensemble totalement ordonné et R_σ un groupe totalement ordonné archimédien pour tout σ, il existe un groupe totalement ordonné maximal $\mathfrak{R}$ l'admettant pour squelette.

Γ est dit régulier si pour tout couple $\Delta^*\subset\Delta$ de sous-groupes isolés successifs on peut trouver un anneau A tel que $\mathbf{Z}\subset A\subset\mathbf{Q}$ et tel que Δ/Δ^* soit un A-module. On a alors la généralisation suivante du théorème de Hahn: Tout groupe régulier peut être plongé dans un groupe maximal ayant même squelette.

Γ est dit algébriquement complet si pour tout sous-groupe isolé Δ de Γ, le groupe Γ/Δ est complet pour la topologie de l'ordre. Un tel groupe est maximal. Γ peut toujours être plongé dans un groupe algébriquement complet $\hat{\Gamma}$ ayant même ensemble P que Γ et dont les $\hat{R}_\sigma$ sont les complétés des R_σ de Γ.

Γ étant quelconque, l'auteur le plonge dans un groupe algébrique complet $\mathfrak{R}$, puis $\mathfrak{R}$ dans un certain monoïde ordonné $\overline{\mathfrak{R}}$ de sorte que se trouve défini un isomorphisme de monoïde ordonné des classes supérieures (surclasses) de Γ dans $\overline{\mathfrak{R}}$. A partir de là l'auteur retrouve les principaux résultats du rapporteur [Bull. Soc. Math. France **80** (1952), 61–100; **81** (1953), 41–61; MR **14**, 613; **15**, 280] sur l'arithmétique des anneaux de valuation et plus généralement des anneaux de Prüfer du type de Dedekind.

P. Jaffard (Paris)

6397:

Artzy, R. On automorphic-inverse properties in loops. Proc. Amer. Math. Soc. **10** (1959), 588–591.

A loop is said to have the inverse property (I.P.) if for each element x of the loop there exists an element x^{-1} such that $x^{-1}(xy)=y=(yx)x^{-1}$ for all y. It has the crossed-inverse property (C.I.) if $xy\cdot xJ=y$ for all y, where J is the mapping which maps every element of the loop on its right inverse. The author shows that an I.P. loop in which J is an anti-automorphism everywhere (i.e., in all of whose isotopes J is an anti-automorphism) is a Moufang loop and that a C.I. loop in which J is an automorphism everywhere is an abelian group.

H. Minc (Vancouver, B.C.)

TOPOLOGICAL GROUPS AND LIE THEORY

6398:

★Fenchel, W. Bemerkungen zur allgemeinen Theorie der diskontinuierlichen Transformationsgruppen. Treizième congrès des mathématiciens scandinaves, tenu à

Helsinki 18–23 août 1957, pp. 77–85. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

This paper contains a simple but useful survey of the relations between various definitions of discontinuity or proper discontinuity of topological transformation groups.

R. C. Gunning (Princeton, N.J.)

6399:

Tôgô, Shigeaki. Note on formal Lie groups. Amer. J. Math. **81** (1959), 632–638.

The author settles several questions raised by J. Dieudonné [Amer. J. Math. **79** (1957), 331–388; MR **20** #931]. We use Dieudonné's notation. Let GL denote the n-dimensional full linear group over an algebraically closed field K. If Γ is an algebraic subgroup of GL, Γ^* denotes the corresponding formal Lie group. If G is a subgroup of the formal group GL^*, $\mathscr{A}(G)$ ($\subset$GL) denotes the algebraic hull of G. Theorem A: Let G and $H \subset G$ be subgroups of GL^*. Then H is a maximal solvable subgroup [maximal torus, Cartan subgroup] of G if, and only if, $\mathscr{A}(H)$ is a maximal solvable connected subgroup [maximal torus, Cartan subgroup] of $\mathscr{A}(G)$. Corollary 1: The centre of a maximal solvable subgroup of G is the centre of G. Corollary 2: Let H be a maximal solvable subgroup, or a Cartan subgroup, or the largest solvable normal subgroup (radical), of G; then $G = \mathscr{A}(G)^*$ if, and only if, $H = \mathscr{A}(H)^*$. Theorem B: Let G be any formal Lie group over an algebraically closed field of characteristic $p > 0$. Then the sup. of the maximal solvable subgroups of G = the sup. of the Cartan subgroups of $G = G$.

G. E. Wall (Sydney)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

6400:

Cohen, Haskell; and Collins, H. S. Affine semigroups. Trans. Amer. Math. Soc. **93** (1959), 97–113.

A convex set S in a topological vector space X—here, usually locally convex—is an "affine topological semigroup" in case S has a continuous multiplication, distributive over convex linear combinations. The authors discuss the structure of such objects, especially in the compact case; typical results are theorem 3: the minimal one-sided ideals are convex, and all points of the kernel are idempotent (the second part being actually an unpublished result of A. L. Shields); and theorem 5: every subgroup containing a non-kernel idempotent is a subset of the boundary of S. Specializing to finite dimensionality the authors discuss representability as semigroups of real matrices (theorem 8). A complete catalogue of one-dimensional affine, and of two-dimensional compact affine, semigroups is given; in all of these the kernel is convex. The latter holds in dimension 3 when compact S has a unit, but, as shown by examples, it does not hold generally.

J. G. Wendel (Ann Arbor, Mich.)

6401:

Anderson, Lee W. One dimensional topological lattices. Proc. Amer. Math. Soc. **10** (1959), 715–720.

The purpose of this paper is to prove that a locally compact, connected, one-dimensional topological lattice is a chain. The proof is based on the fact that a locally compact, connected, topological lattice is locally convex.

L. Nachbin (Rio de Janeiro)

FUNCTIONS OF REAL VARIABLES

See also 6418, 6686.

6402:

Levine, Norman. A note on functions continuous almost everywhere. Amer. Math. Monthly **66** (1959), 791–792.

A function is said to be C.A.E. if it is continuous almost everywhere. The following theorem is proved: $f: R \to R$ is C.A.E. if and only if, for O^* open, the set $f^{-1}(O^*)$ can be written as a union $O \cup A$, where O is open and the Lebesgue measure of A is zero. The author concludes with some remarks on conditions under which the function fg is C.A.E. when both f and g are C.A.E.

U. S. Haslam-Jones (Oxford)

6403:

Marcus, S. Sur les fonctions continues qui ne sont monotones en aucun intervalle. Rev. Math. Pures Appl. **3** (1958), 101–105.

K. Padmavally [Proc. Amer. Math. Soc. **4** (1953), 839–841; MR **15**, 513] has proved that if $f(x)$ is real and continuous on a one-dimensional interval I and $f(x)$ is not monotone on any interval $J \subset I$ then the set of values of y such that $f(x) = y$ in a set of the power of the continuum is a set of the second category. The author proves further that a necessary and sufficient condition that a continuous $f(x)$ be not monotone in any $J \subset I$ is that for each such J the set of values of y such that $f(x) = y$ in a non-enumerable subset of J forms a set of the second category on $(-\infty, \infty)$ which is residual on $f(J)$. He then discusses further the theorem of Padmavally, and a theorem due to S. Minakshisundaram [J. Indian Math. Soc. **4** (1940), 31–33; MR **1**, 303] concerning non-differentiable functions.

U. S. Haslam-Jones (Oxford)

6404:

Iosifescu, Marius. Sur les fonctions continues dont les ensembles de niveau sont au plus dénombrables. Rev. Math. Pures Appl. **3** (1958), 439–441.

A. Marchaud [Fund. Math. **20** (1933), 105–116] has proved that any real function of a real variable which takes on each value at most a finite number of times is differentiable almost everywhere. The author shows that a function which takes on each value in an at most enumerable set may be differentiable only in a set of arbitrarily small measure.

U. S. Haslam-Jones (Oxford)

6405:

Dombrowski, Peter. Über punktal-dehnungsbeschränkte und (total) differenzierbare Abbildungen. Arch. Math. **10** (1959), 144–150.

The author considers a mapping f of R^m in R^n which is "punktal-dehnungsbeschränkt" and is defined in a region $Df \subset R^m$. (f is called dehnungsbeschränkt if there exists $\kappa > 0$ such that, whenever $p \in Df$ and $q \in Df$, $\sigma(f(p), f(q)) \leqq \kappa\rho(p, q)$, where ρ is the metric in R^m and σ is the metric in R^n: f is called punktal-dehnungsbeschränkt at a point $p \in Df$ if there exist positive numbers ε, κ such that, for all $q \in Df$, $\rho(p, q) \leqq \varepsilon$ implies $\sigma(f(p), f(q)) < \kappa\rho(p, q)$.) He so obtains more general forms of theorems due to Rademacher, Stepanoff and others [H. Rademacher, Math. Ann. **79** (1919), 340–359; W. Stepanoff, Mat. Sb. **32** (1924), 511–526]. One result is: Let f be a λ_m-measurable

mapping of R^m in R^n (where λ_m denotes Lebesgue measure in R^m) with $Df=X$ a λ_m-measurable set. Let D be the subset of X in which f is totally differentiable, E the subset of X in which f is punktal-dehnungsbeschränkt. Then (a) $D\subset E$ and $\lambda_m(E-D)=0$, (b) f has on D a λ_m-measurable differential. *U. S. Haslam-Jones* (Oxford)

6406:

Besicovitch, A. S.; et Schoenberg, I. J. Sur les arcs ascendants à pente partout nulle et des problèmes qui s'y rattachent. C. R. Acad. Sci. Paris **249** (1959), 1079–1080.

Let $x=f(t)$ be a Jordan arc in real Euclidean or Hilbert space E, and let $g(t)$ be a continuous function, $0\leq t\leq 1$. If

$$\lim_{h\to 0}\frac{g(t+h)-g(t)}{\|f(t+h)-f(t)\|}=0$$

for all t, does it follow that $g(t)$ is constant? Yes, if E is a line; no, if E is Hilbert space or the plane. The latter conclusion results from the construction of a plane arc J with the property that to each $\varepsilon>0$ corresponds a constant C_ε such that the plane measure of the subarc with extremities ν and ν' is positive and less than $C_\varepsilon|\nu-\nu'|^{2-\varepsilon}$. This condition cannot be satisfied for $\varepsilon=0$. Two related results are: If $x=f(t)$ is a plane arc then $\liminf\sum_1^n|f(t_i)-f(t_{i-1})|^2=0$, where the lim inf is taken over all subdivisions $0=t_0<t_1<\cdots<t_n=1$ as $\max|t_i-t_{i-1}|\to 0$. A corollary is that if $g(t)$ satisfies a Lipschitz condition $|g(t)-g(t')|\leq C|f(t)-f(t')|^2$ then $g(t)$ is constant.

J. C. Oxtoby (Bryn Mawr, Pa.)

6407:

Császár, Ákos. Sur les ensembles et les fonctions convexes. Mat. Lapok **9** (1958), 273–282. (Hungarian. Russian and French summaries)

Let $f(\frac{1}{2}(x+y))\leq\frac{1}{2}f(x)+\frac{1}{2}f(y)$ on a convex set in n-dim. space. If $g(x)$ is measurable and $f(x)\leq g(x)$ on a subset of positive measure then $f(x)$ is continuous. (Previously known in dimension 1.) Also

$$\underline{f}(x)=\lim_{\delta\to 0}\inf\{f(t):|t-x|<\delta\}$$

is either a continuous convex function or everywhere $=-\infty$. *P. Ungar* (New York, N.Y.)

6408:

Sharma, A. Some properties of plane curves. Ann. Polon. Math. **6** (1959), 245–252.

The author gives limiting ratios of differences of areas bounded by a planar curve and its chords to other areal differences on the assumption of certain order properties of the function the curve represents. These results extend similar ones of S. Gołąb [same Ann. **1** (1954), 91–106; MR **16**, 67]. Quadrature formulas can be developed from these results, extending the classical theorem of Archimedes that the area of a circumscribing rectangle based on a chord of a parabola is divided in the ratio 1:2 by the arc of the parabola. *P. C. Hammer* (Madison, Wis.)

6409:

Lagrange, Jean. Sur les intégrales

$$I_n=\int_{-\infty}^{+\infty}\frac{\sin^n x}{x^n}f(x)dx,$$

n entier positif, $f(x+\pi)=\pm f(x)$. Mathesis **68** (1959), 42–46.

The author uses the Euler formulae for $\cot x$ and $1/\sin x$ to reduce the integral given in the title to an integral over the segment $(0,\pi)$; and $(0,\frac{1}{2}\pi)$, whenever $f(x)$ is even. Several applications are made for the cases $n=1, 2, 3$. For instance, if $f(x)=|\sin^3 x|$ then I_1 becomes the integral of Wallis. *W. C. Royster* (Lexington, Ky.)

MEASURE AND INTEGRATION

6410:

Gaşpar, Toma. Théorème sur l'intégrabilité absolue des fonctions intégrables au sens généralisé. Com. Acad. R. P. Romîne **9** (1959), 305–308. (Romanian. Russian and French summaries)

In his text, *Analiza matematica* [Acad. R. P. Romîne, Bucharest, 1953], Nicolescu proved for a generalization of the Riemann integral to not-necessarily-bounded functions on not-necessarily-bounded domains most of the properties of the ordinary Riemann integral. This note gives a proof that the absolute value of an integrable function is integrable. *M. M. Day* (Urbana, Ill.)

6411:

Schoenberg, I. J. The integrability of certain functions and related summability methods. II. Amer. Math. Monthly **66** (1959), 562–563.

The author makes three remarks to his previous paper [same Monthly **66** (1959), 361–375; MR **21** #3696]. He gives a simple proof of Besicovitch of the fact that to every δ there exists an n_0 so that for every $n>n_0$ each subinterval I_δ of length δ contains a fraction m/n, $(m,n)=1$. He also uses a result of P. Hartman [Quart. J. Math. Oxford Ser. **18** (1947), 124–127; MR **9**, 137] to generalize a result of his paper. *P. Erdös* (Adelaide)

6412:

Sudakov, V. N. Linear sets with quasi-invariant measure. Dokl. Akad. Nauk SSSR **127** (1959), 524–525. (Russian)

Let E be a linear space, B a σ-algebra of subsets of E invariant with respect to translations and homotheties, and μ a finite measure on B; μ is said to be quasi-invariant if $\mu(x+A)=0$ whenever $\mu(A)=0$ and $x\in E$. It is proved that under certain conditions (for instance, if E is locally convex and B contains all open sets), if μ is quasi-invariant, then E is finite-dimensional. *N. Dinculeanu* (Bucharest)

6413:

Rogers, C. A.; and Taylor, S. J. The analysis of additive set functions in Euclidean space. Acta Math. **101** (1959), 273–302.

The authors extend the classical resolution of a completely additive set function F defined over the Borel subsets of a closed interval of real n-dimensional Euclidean space, by considering (a) a general Hausdorff measure in place of Lebesgue measure and (b) a class of comparable

Hausdorff measures, in particular the class of fractional measures. Typical results are the following.

(a) Let $h(t)$ be a Hausdorff measure function and denote the h-measure of E by $\Lambda_h(E)$. Then F can be uniquely expressed as the sum of three completely additive set functions F_1, F_2, F_3, where (i) $F_1(E)=0$ for every set E of σ-finite h-measure, (ii) either F_2 is identically zero or $F_2(E)=\int_{E\cap S} f(x)d\Lambda_h$, where S is a Borel set of positive σ-finite h-measure and $f(x)\neq 0$, $x\in S$, (iii) there is a set K with $\Lambda_h(K)=0$ and $F_3(E)=F_3(E\cap K)$.

(b) For each completely additive set function F there is a finite or enumerable sequence of real numbers $\alpha_1, \alpha_2, \cdots$, where $\alpha_i\neq\alpha_j$ if $i\neq j$ and $0\leq\alpha_i\leq n$, such that

$$F = F^{(d)} + F^{(\alpha_1)} + F^{(\alpha_2)} + \cdots,$$

where (i) if $F^{(d)}$ is decomposed as in (a) above for a Hausdorff measure function t^α, $0\leq\alpha\leq n$, then the component F_2 is identically zero, (ii) $F^{(\alpha_i)}(E)=0$ for every Borel set E of dimension less than α_i and there exists a set K of dimension α_i such that $F^{(\alpha_i)}(E)=F^{(\alpha_i)}(E\cap K)$.

An example is given of a set function for which $F^{(d)}$ is not identically zero. Extensions of the results of (b) to a maximal class of comparable Hausdorff measure functions are given. The paper concludes with a number of remarks and conjectures as to further possible extensions of these results. *H. G. Eggleston* (London)

6414:

Zítek, František. Über die Burkillschen Integrale, die von einem Parameter abhängen. Časopis Pěst. Mat. **84** (1959), 165–176. (Czech. Russian and German summaries)

Using the concept of uniform convergence, the author proves some theorems on the one-dimensional Burkill integral, which are analogous to theorems of the Riemann or Lebesgue theory of integration. Theorems 2–5 are concerned with continuity with respect to a parameter; theorem 6, the limit; theorem 7, differentiation; and theorems 8 and 9, integration. Theorems 10–12 provide examples of an application of the previously stated theorems to Riemann and Lebesgue integrals.

J. Janko (Prague)

FUNCTIONS OF A COMPLEX VARIABLE

6415:

★Hille, Einar. Analytic function theory. Vol. 1. Introduction to Higher Mathematics. Ginn and Company, Boston, 1959. xi+308 pp. $6.50.

Written in a pleasant and unhurried style, this book departs in many respects from currently held ideas as to what a college text should be. The author neither accepts the narrow utilitarian outlook of those who are convinced of the superiority of vegetables over flowers, nor does he have much sympathy for the true believer who feels that anything written, say, more than twenty years ago belongs to the heathen past and should be relegated to oblivion lest it corrupt the impressionable young. While all the standard topics are covered, the presentation is often guided by aesthetic and, especially, historical considerations. There are a large number of historical footnotes supplying biographical details of many major—and also minor—contributors to the theory of functions, and the author makes every possible effort to communicate to the reader his sense of the historical continuity of the subject.

The chapter headings of the present volume are: (1) Number systems. (2) The complex plane. (3) Fractions, powers, and roots. (4) Holomorphic functions. (5) Power series. (6) Some elementary functions. (7) Complex integration. (8) Representation theorems. (9) The calculus of residues. There are also three appendices, containing supplementary material on point sets, polygons, and Riemann-Stieltjes integration. A second volume, covering more advanced topics, is to follow fairly soon.

Z. Nehari (Pittsburgh, Pa.)

6416:

Elyash, E. S.; and Levine, Norman. A note on the function $W=Az+B$. Amer. Math. Monthly **66** (1959), 803.

The authors prove by elementary considerations that the linear functions are the only analytic (conformal) convex functions f satisfying functional equations of the form (*) $f((1-p)z_1+pz_2)=(1-q)f(z_1)+qf(z_2)$, z_1, z_2 being arbitrary complex values and p, q being real and possibly depending upon z_1, z_2. (The paper states (*) in another equivalent form.) The converse is evident.

J. Aczél (Debrecen)

6417:

Goldman, A. J. Zeros of certain polynomials. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 21–22.

Let n be a fixed non-negative integer. The author poses the following problem. Find an explicit characterization of the set S_n of all values of the real parameter P for which all roots of $z^{n+1}-z^n+P=0$ lie in the disk $|z|<1$. His result: $S_n=\{P\mid 0<P<2\sin\pi/(4n+2)\}$.

V. F. Cowling (Lexington, Ky.)

6418:

Schoenfeld, Lowell. On integrals of the type

$$\int_0^\infty f(x)\log\left|\frac{x+1}{x-1}\right|dx.$$

SIAM Rev. **1** (1959), 154–157.

If $f(z)$ is real and continuous and an odd function on the real axis; is analytic in the upper half plane except for a finite number of poles $z_1, \cdots, z_n$; and for some real c and for large z, $f(z)=c/z+o(1/z\log|z|)$ uniformly in the closed upper half plane; then by the use of contour integration it is shown that

$$\int_0^\infty f(x)\log\left|\frac{x+1}{x-1}\right|dx = \frac{1}{2}\pi^2 c - 2\pi\mathrm{I}\left(\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}(f(z)\log(z+1))\right).$$

The continuity condition on f on the real axis can be relaxed so as to permit discontinuities at a finite number of points $\xi_1, \cdots, \xi_m$ at each of which $f(z)$ behaves like $(z-\xi_i)^{\alpha_i}F(z)$, $\alpha_i>-1$, $F(z)$ continuous.

T. H. Hildebrandt (Chapel Hill, N.C.)

6419:

Alpár, László. Remarque sur la sommabilité des séries de Taylor sur leurs cercles de convergence. II. Magyar

Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 141–158. (Hungarian and Russian summaries)

This is a continuation of earlier work by the author [same Közl. **3** (1958), 1–12; MR **21** #1382] and by Turán [Akad. Serbe Sci. Publ. Inst. Math. **21** (1958), 19–26; MR **12** #1381]. Let Δ be the unit disc and ϕ a conformal map of Δ onto itself. Let f be any function analytic in Δ, and let $g(z)=f(\phi(z))$. Previous work showed that if the power series for f is Abel summable at t_0, that for g is Abel summable at $\phi^{-1}(t_0)$, but that this failed in general if Abel summability is replaced by (C, k), $k \geqq 0$. The author now proves the very interesting fact that if the series for f is summable (C, k) at t_0, that for g is always summable $(C, k+1)$ at $\phi^{-1}(t_0)$. *R. C. Buck* (Princeton, N.J.)

6420:

Rogozhin, V. S. Sufficient conditions for the existence of unique solutions of inverse boundary value problems. J. Appl. Math. Mech. **22** (1958), 1141–1148 (804–807 Prikl. Mat. Meh.).

The author derives sufficient conditions on $p(\theta)$ for the mapping defined by

$$\frac{dz}{d\zeta} = \exp\left[-\frac{1}{2\pi}\int_0^{2\pi} p(\theta)\,\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\,d\theta\right]$$

to be univalent on the exterior of the unit circle. (The word "univalent" is translated as "unique" throughout.) This problem arises in determining an airfoil profile from the velocity distribution.

A similar problem arising in the theory of filtration is also treated. *H. F. Weinberger* (College Park, Md.)

6421:

★**von Koppenfels, Werner; und Stallmann, Friedemann. Praxis der konformen Abbildung.** Die Grundlehren der mathematischen Wissenschaften, Bd. 100. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. xiii + 375 pp. DM 69.00.

An extensive collection on conformal mapping was planned by a German circle of workers in function theory. As its first part the senior author completed his manuscript "Systematik der Polygonalabbildung". But the post-bellum confusion as well as his death in 1945 interrupted its publication for a long time. The manuscript was subsequently transferred to the junior author who supplemented it extensively, transforming it into a textbook-like form, an essential change from the original plan. The book consists of two parts, theory of conformal mapping and catalogue of conformal mapping, of which the first half of the second part is the posthumous work of the senior author.

The first part consists of five chapters. The first chapter contains preparatory sections explaining complex numbers and functions in their geometrical meaning. The second chapter deals with conformal mapping by several elementary functions. The third chapter presents some basic theorems for potential and function theory. Boundary value problems for functions harmonic in a circle are discussed. Further, Riemann's mapping theorem is explained without proof. The fourth and principal chapter discusses especially in detail the theory and practise of mapping polygons onto a half-plane. The fifth chapter is concerned with approximation processes of conformal mapping which will suggest in part also a way to prove the mapping theorem.

The second part is devoted to a systematic survey of such domains which can be dealt with by means of the methods presented in the first part. Polygonal mappings play here also the main role. Many domains associated with elementary or elliptic mapping functions are enumerated systematically. In every case the endeavor is made to use as many as possible parameters characterizing the domains, in order to ensure their generality and effectiveness from the practical point of view.

Designed for workers with a good current knowledge of advanced calculus, the book deals with the subjects ranging from introductory general theory to some modern development. Its aim is to make the reader independent of special literature, so that the book refers to no original papers except those essentially outside its own field. It seems also very natural that in a book of this character the full proof of every statement must be conceded to a certain extent. Many excellent figures are inserted for illustration throughout. The book will be very useful both as an introduction to and as a dictionary of conformal mapping. *Y. Komatu* (Tokyo)

6422:

Erohin, V. On the theory of conformal and quasiconformal mapping of multiply connected regions. Dokl. Akad. Nauk SSSR **127** (1959), 1155–1157. (Russian)

Announcement of results concerning the factorization (in the sense of composition) of univalent conformal and quasi-conformal maps of plane regions of finite connectivity. A number of applications are also stated. The principal theorem for the conformal case is the following. Let D denote an arbitrary n-tuply connected region of the extended plane. Let $\zeta=f(z)$ denote an arbitrary univalent conformal map of D. Then there exist n conformal maps $\zeta_k=f_k(\zeta_{k-1})$ $(k=1, \cdots, n;\ \zeta_0=z,\ \zeta_n=\zeta)$ of certain simply-connected regions such that f may be represented by superposition of the f_k as follows:

$$(*) \qquad f(z) = f_n(f_{n-1}(\cdots(f_1(z))\cdots)).$$

If $C_1, C_2, \cdots, C_n$ is a given indexing of the components of the boundary of D, the map f_k may be chosen to be a univalent conformal map of the simply-connected region $D_{k-1,k}$ containing the domain $D_{k-1}=f_{k-1}(\cdots(f_1(D))\cdots)$ and having as boundary $C_{k-1,k}=f_{k-1}(\cdots(f_1(C_k))\cdots)$ $(C_{0,1}=C_1,\ D_0=D)$. Subject to this restriction the factorization (*) is unique up to intermediate linear fractional transformations. This theorem has a counterpart for quasi-conformal maps. An application that is given of the quoted theorem is the following. Let $S_r=\{|z|<r\}$, $0<r\leqq 1$, and let f be a bounded analytic function on S_1; if cap $(\overline{f(S_1)})\leqq 1$, then cap $(\overline{f(S_r)})\leqq r$, $0<r<1$.

M. H. Heins (Urbana, Ill.)

6423:

Hara, I. S. Some approximate formulas in the theory of conformal mappings. Dokl. Akad. Nauk SSSR **126** (1959), 1210–1213. (Russian)

Approximation methods are given for the numerical use of the Schwarz-Christoffel formulae in the case of certain special classes of polygons.

M. H. Heins (Urbana, Ill.)

6424:

Krzyż, J. Circular symmetrization and Green's function. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 327–330. (Russian summary, unbound insert)

The author proves that if D is a simply connected domain and D^* denotes the domain obtained from D by symmetrization with respect to a line or ray passing through two points P_1, P_2, then the Green's function g satisfies

$$(*) \qquad g(P_1, P_2; D^*) \geqq g(P_1, P_2; D).$$

The result is deduced by integration from the case when the distance P_1P_2 is infinitesimal, in which case (*) is equivalent to Pólya's corresponding theorem for the inner radius at P_1. *W. K. Hayman* (London)

6425:

Fox, William C. A new inequality for the Green's function. Proc. Amer. Math. Soc. **10** (1959), 562–569.

The author claims to improve Koebe's original proof of his distortion theorem [Nachr. K. Gesell. Wiss. Göttingen Math.-Phys. Kl. **1907**, 191–210; p. 203] in such a way that it gives directly the right constant, $\frac{1}{4}$. Unfortunately, this improvement fails to be correct: the inequality in lemma 3.1. has to be written in the opposite sense as follows immediately by the fact that Green's function is an increasing functional of the domain.

A. Pfluger (Zürich)

6426:

Fox, William C. Sufficient conditions that a Peano-interior function have an extension with constant boundary values. Trans. Amer. Math. Soc. **90** (1959), 255–271.

Soit M une variété à deux dimensions sur laquelle est donnée une fonction f Peano-intérieure (c'est à dire une fonction continue sur M, intérieure sur $\mathring{M}$, et dont les courbes de niveau sont localement connexes en tout point $\in \mathring{M}$). Dans un article précédent [Trans. Amer. Math. Soc. **83** (1956), 338–370; MR **18**, 885] l'auteur avait montré que dans le cas où M est compacte et où chaque courbe frontière de M est une courbe de niveau de f, la somme des multiplicités des points critiques de f est égale au premier nombre de Betti de M, diminué de 1.

L'objet du présent mémoire est de déterminer la somme des multiplicités des points critiques pour le cas où M compacte n'est plus limitée par des courbes de niveau de f. Pour cela l'auteur cherche à réaliser une extension $C(M)$ de M (par l'adjonction d'anneaux ou de bandes) et à déterminer M^* ($\mathring{M} \subset M^* \subset C(M)$), limitée par des courbes de niveau d'une fonction f^*, extension de f, Peano-intérieure sur M^*. Il suffira alors d'évaluer la somme des multiplicités des points critiques dans $M^* - \mathring{M}$ pour obtenir le nombre cherché.

Le problème essentiel devient celui de la construction de f^*. Les conditions sous lesquelles l'auteur parvient à réaliser cette extension s'expriment en termes de S-régularité: Soit J une courbe frontière de M; J est S-régulière $\Leftrightarrow$ pour tout $p \in J$ l'ensemble $\{x | x \in \mathring{M}, f(x) = f(p)\} \cup \{x | x \in J, f(x) > f(p)\}$ est localement connexe en p. En remplaçant $f(x) > f(p)$ par $f(x) < f(p)$ on définit la L-régularité de J.

Le dénombrement des multiplicités des points critiques est alors réalisé dans le cas où M est compact et où l'on peut démontrer la S-régularité (ou la L-régularité) des courbes frontières.

Si toutes les courbes frontières sont S-régulières et si f est pseudo-harmonique sur $\mathring{M}$, l'extension de f en f^* sur M^* est aussi pseudo-harmonique.

Dans le cas où f est harmonique, deux problèmes sont alors posés concernant la possibilité d'introduire sur M^* un système de coordonnées tel que l'extension de f soit harmonique sur M^* (pour ce système de coordonnées). L'auteur indique les principales raisons des difficultés rencontrées dans l'étude de ces problèmes.

L. Fourès (Marseille)

6427:

Oğuztöreli, M. Namık. Sur les cas d'épuisement des surfaces de Riemann à type hyperbolique. Rev. Fac. Sci. Univ. Istanbul. Sér. A **22** (1957), 123–126. (Turkish summary)

The author extends the criterion of L. V. Ahlfors [Acta Math. **65** (1935), 157–194; p. 187] for a simply-connected hyperbolic Riemann surface to be "regulär ausschöpfbar" to multiply connected Riemann surfaces. The sequence of exhausting subsurfaces F_r consists of Riemannian domains bounded by suitable level curves of a Green's function. *W. K. Hayman* (London)

6428:

Kuramochi, Zenjiro. Representation of Riemann surfaces. Osaka Math. J. **11** (1959), 71–82.

A number of covering properties are obtained for a meromorphic function whose domain is a Riemann surface of certain distinguished classes or is a subregion of a certain kind of such a surface. The following result is typical. Let R denote a Riemann surface belonging to the class HND ($0 \leqq N \leqq \infty$) consisting of the Riemann surfaces on which there are at most N linearly independent Dirichlet bounded harmonic functions. Let G denote a non-compact region of R. Let f denote an analytic function on G. Let $K = \{|W - W_0| < r\}$ and let Ω denote a component of $f^{-1}(K)$. Let the Riemannian image of Ω with respect to f have infinite area. Let $n(w)$ denote the number of times (counted according to multiplicity) that f attains w in Ω. Let $D_n = \{w | n(w) \geqq n\}$. Then $\bar{D}_\infty = \bigcap \bar{D}_n \neq \emptyset$. Let $\Omega_1, \Omega_2, \cdots$ denote the components of $K - \bar{D}_\infty$. Then $n(w) = \sup_{\Omega_i} n(w)$ in Ω_i save for a closed set of zero capacity in Ω_i. *M. H. Heins* (Urbana, Ill.)

6429:

Matsumoto, Kikuji. On subsurfaces of some Riemann surfaces. Nagoya Math. J. **15** (1959), 261–274.

This paper gives proofs of some theorems of Kuramochi [Osaka Math. J. **7** (1955), 109–127; **10** (1958), 83–102; MR **17**, 26; **20** #3272] with the aid of methods developed by C. Constantinescu and A. Cornea [Nagoya Math. J. **13** (1958), 166–233; MR **20** #3273]. The theorems in question pertain to the classification problem for Riemann surfaces and state roughly that, if the surface itself belongs to a certain class as far as the behavior of bounded [Dirichlet bounded] harmonic functions is concerned, i.e., if the sur-

face belongs to the class $O_{HB_n}-O_G$ [$O_{HD_n}-O_G$], $1 \leqq n \leqq \infty$, then certain subregions satisfying conditions involving the existence of certain bounded [Dirichlet bounded] harmonic functions belong to the class O_L (no non-constant Lindelöfian meromorphic functions) [O_{AD}]. Covering properties of meromorphic functions on a surface of class O_{HB_n} [O_{HD_n}] are obtained. *M. H. Heins* (Urbana, Ill.)

6430:

Teleman, C. Sur les structures homographiques d'une surface de Riemann. Comment. Math. Helv. **33** (1959), 206–211.

This paper generalizes an earlier result of the author [Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 163–182; MR **20** #1772]. Let R denote a Riemann surface. Let $(u_1, \cdots, u_{p+1})$ denote analytic elements continuable on R. It is supposed that: (1) if $(u_1, \cdots, u_{p+1})$ is continued along a closed path of R into $(\bar{u}_1, \cdots, \bar{u}_{p+1})$, then

$$(*) \qquad \rho\bar{u}_i = \sum_1^{p+1} c_i^j u_j \quad (i = 1, \cdots, p+1),$$

where the c_i^j are constants and ρ is analytic; (2) the Wronskian $\det(u_i^{(p+1-j)})$ is nowhere zero (the derivatives being taken with respect to a local uniformizer); (3) the u_i have no other singularities than poles. Two such systems $(u_1, \cdots, u_{p+1})$ and $(v_1, \cdots, v_{p+1})$ are termed equivalent provided that they satisfy relations of the form (*) in a simply-connected region $\subset R$. The equivalence classes are termed homographic structures of order p. A (1, 1) correspondence is set up between the structures of order p and the p-tuples $(\omega_2, \cdots, \omega_{p+1})$ where ω_k is an analytic differential of order k. The special case $p=2$ was treated by the author in the paper cited above.

M. H. Heins (Urbana, Ill.)

6431:

McLeod, Robert M. On the zeros of the derivatives of some entire functions. Trans. Amer. Math. Soc. **91** (1959), 354–367.

Let f be an entire function, let E_k be the set of zeros of $f^{(k)}(z)$ and let L_f be the set of all z which either belong to infinitely many E_k or are limit points of $\bigcup(E_k)$. The author considers entire functions of the form $f(z) = \exp(cz^q + dz^{q-1})\phi(z)$, where $q \geqq 2$, and proves that, under certain conditions on $\phi(z)$, L_f consists of the set R_f of q rays emanating from the point $-d/(qc)$ and making angles $(\theta + (2k+1)\pi)/q$ with the real axis where $\theta = -\arg c$ and k runs from 0 to $q-1$.

This result had been proved by Pólya [Math. Z. **12** (1922), 36–60] if $\phi(z)$ has order less than $q-1$ and only a finite number of zeros. The author weakens these conditions considerably. He shows, for instance, that it is sufficient to assume that each of the bisectors of the angles between the rays of R_f lies in an angle which contains only a finite number of zeros of $\phi(z)$, and in which $\phi(z)$ has at most order $q-1$ minimal type, and that outside these angles $\phi(z)$ has at at most order q minimal type.

Proofs are fairly lengthy and technical. By extending a technique of the reviewer [J. Reine Angew. Math. **196** (1956), 67–95; MR **18**, 293] the author obtains an asymptotic formula for $f^{(n)}(z)$ as n tends to infinity while z lies in a fixed disc whose closure does not meet R_f. In particular, $f^{(n)}(z)$ has no zeros in such a disc for large n, so that the interior points of the disc cannot be points of L_f. Thus L_f is a sub-set of R_f. Further, the approximating functions for $f^{(n)}(z)$ change in a rather violent way as z crosses one of the rays of R_f, and from this and a theorem of Ganelius [Ark. Mat. **3** (1954), 1–50; MR **16**, 23] the author is able to conclude that, in a fixed disc centred on a point of R_f, $f^{(n)}(z)$ has at least $Kn^{(1-1/q)}$ zeros for large n, so that R_f is a sub-set of L_f. *W. K. Hayman* (London)

6432:

Srivastav, R. P. On zeros, poles and mean value of meromorphic functions. Compositio Math. **13**, 219–228 (1958).

Es sei $f(z)$ meromorph von der endlichen Ordnung $\rho > 0$ in $|z| < \infty$. In Abschnitt I werden durch elementare Abschätzungen verschiedene Ungleichungen zwischen lim sup und lim inf der Anzahlfunktionen $n(r)/r^\rho$ und $N(r)/r^\rho$ hergeleitet, ebenso eine Abschätzung der Änderung der charakteristischen Funktion $T(r)$. In Abschnitt II werden folgende Sätze bewiesen. (1) Es sei $f(z)$ meromorph von der Ordnung $\rho < 1$, alle Nullstellen und Pole seien negativ reell, für die Anzahlfunktionen gelte ferner asymptotisch $n(t, 0) - n(t, \infty) \sim \lambda t^\rho$. Dann gilt auf dem Strahl $z = te^{i\theta}$ ($|\theta| < \pi$) asymptotisch

$$f'(z)/f(z) \sim \lambda \cdot e^{i(\rho-1)\theta} z^{\rho-1} \cdot \pi\rho \operatorname{cosec} \pi\rho.$$

(2) Es sei $f(z)$ von der Ordnung $\rho < 2$, $f(z)$ reell für z reell, alle Nullstellen a_n und Pole b_n seien positiv reell, nach wachsendem Betrag numeriert und es sei $a_n < b_n$. Dann liegen alle Nullstellen von $f'(z)$ in $\operatorname{Re} z < \frac{1}{2}(a_1 + b_1)$. (3) Es habe $f(z) = z^3 + 3Hz + G$ nur reelle Nullstellen; $\varphi(w)$ sei meromorph vom Geschlecht Null oder Eins, sei reell für reelle w und habe nur reelle Nullstellen $-\alpha_n$ und Pole $-\beta_n$ mit $\alpha_n < \beta_n$. Dann hat $g(z) = \varphi(3)z^3 + 3H\varphi(1)z + G\varphi(0)$ nur reelle Nullstellen. Im Abschnitt III ist kaum zu folgen. *F. Huckemann* (Giessen)

6433:

Chamfy, Christiane. Fonctions méromorphes dans le cercle-unité et leurs séries de Taylor. Ann. Inst. Fourier. Grenoble **8** (1958), 211–262.

Es sei $\sum_0^\infty u_n z^n$ die Nullpunktsentwicklung der meromorphen Funktion $f(z)$. Verf. beweist: Es gibt rationale Ungleichungen in den Koeffizienten u_n, die notwendig und hinreichend dafür sind, dass $|f(z)| \leq 1$ ist für $|z| = 1$ und genau p von 0 verschiedene Pole in $|z| < 1$ hat. Ist überdies $f(z)$ reell für reelles z, so sei der Rang s von $f(z)$ endlich, wenn $f(z) = d(z)/e(z)$ und $d(z) = \pm z^s e(1/z)$ ein Polynom vom Grad s ist, sonst sei $s = \infty$. Über die rationalen Ungleichungen gilt der Satz: (1) Für jedes n mit $n_0 \leqq n \leqq s$ (n_0 abhängig von f) gibt es eindeutig bestimmte teilerfremde Polynome $d_n^+(z)$ und $e_n^+(z)$ und teilerfremde Polynome $d_n^-(z)$ und $e_n^-(z)$ vom Grad n mit folgenden Eigenschaften: (i) $e_n^+(0) = 1$, $e_n^-(0) = -1$, $d_n^+(z) = z^n e_n^+(1/z)$, $d_n^-(z) = -z^n e_n^-(1/z)$; (ii) $e_n^+(z)$ und $e_n^-(z)$ haben genau p Nullstellen in $|z| < 1$; (iii) $d_n^+(z)/e_n^+(z) = u_0 + \cdots + u_{n-1}z^{n-1} + \nu_n^+ z^n + \cdots$, $d_n^-(z)/e_n^-(z) = u_0 + \cdots + u_{n-1}z^{n-1} + \nu_n^- z^n + \cdots$. (2) Es gilt $\nu_n^- < u_n < \nu_n^+$ für $n_0 \leqq n < s$, $\nu_n^- < u_s = \nu_s^+$ wenn $f(1) = +1$, $\nu_n^- = u_s < \nu_s^+$ wenn $f(1) = -1$.

Es folgen Sätze über im Einheitskreis meromorphe Funktionen $f(z)$, deren Reihenentwicklung im Nullpunkt nur ganzzahlige Koeffizienten hat: $f(z)$ ist rational, wenn

$f(z)-a$ für irgendeine komplexe Zahl a in einem verallgemeinerten Sinn nur endlich viele Nullstellen im Einheitskreis hat. Hat $f(z)$ höchstens zwei Pole in $|z|<1$, so ist $f(z)$ nicht beschränkt in der Umgebung von $|z|=1$, wenn diese Pole einen grösseren Betrag als θ_1 haben, wobei $\theta_1^3-\theta_1-1=0$ und $0<\theta_1<1$ ist; dies Ergebnis ist scharf. *F. Huckemann* (Giessen)

6434:

Tanaka, Chuji. Note on the cluster sets of the meromorphic functions. Proc. Japan Acad. **35** (1959), 167–168.

Let the function $f(z)$ be meromorphic in the region D whose boundary is C, and suppose that z_0 is a non-isolated boundary point of D. Denote by $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$, $R_{z_0}^{(D)}$, respectively, the interior cluster set, the boundary cluster set, the range of values of $f(z)$ at z_0. The open set $S_{z_0}^{(D)}-S_{z_0}^{(C)}$ is called G; the set of interior points of $S_{z_0}^{(D)}$ is called $O_{z_0}^{(D)}$. It was shown by Noshiro [Jap. J. Math. **17** (1940), 37–96; MR **2**, 81; p. 84] that if G is not empty then $R_{z_0}^{(D)}$ is everywhere dense in G. Extending this result, the author proves, by an argument of Bagemihl [Math. Z. **67** (1957), 49–50; MR **19**, 129], that if $O_{z_0}^{(D)}$ is not empty then $R_{z_0}^{(D)}$ contains an everywhere dense G_δ subset of $O_{z_0}^{(D)}$. A corollary states that if $R_{z_0}^{(D)}$ is empty then $S_{z_0}^{(D)}$ has no interior point.

F. Bagemihl (South Bend, Ind.)

6435:

Lovator, A. Dž. [Lohwater, A. J.] On the theorems of Gross and Iversen. Dokl. Akad. Nauk SSSR **126** (1959), 707–708. (Russian)

The author generalizes the following theorem of Gross and Iversen. If f is meromorphic in the unit disk D, then for each value θ $(0\leq\theta<2\pi)$ and for each value w in $C(f, e^{i\theta})-C_B(f, e^{i\theta})$, with at most two exceptions, each neighborhood of $e^{i\theta}$ relative to D contains a point z for which $f(z)=w$; if an exceptional value exists, it is an asymptotic value for f at $e^{i\theta}$. Here $C(f, e^{i\theta})$ denotes the ordinary cluster set, and C_B denotes the boundary cluster set:

$$C_B(f, e^{i\theta}) = \bigcap_{\eta>0}\left[\left\{\bigcup_{0<|\phi-\theta|<\eta} C(f, e^{i\phi})\right\}^*\right],$$

where the asterisk indicates closure. To obtain a theorem which is significant, for example, with reference to the class of Blaschke products without regular points on $|z|=1$, the author replaces the boundary cluster set by the radial boundary cluster set relative to a set E:

$$C_{R\text{-}E}(f, e^{i\theta}) = \bigcap_{\eta>0}\left[\left\{\bigcup_{\substack{0<|\phi-\theta|<\eta \\ e^{i\phi}\in E}} C_p(f, e^{i\phi})\right\}^*\right].$$

Theorems: If f is meromorphic in D, and E is a set of measure 0 on $|z|=1$, then each value in $C(f, e^{i\theta})-C_{R\text{-}E}(f, e^{i\theta})$, with the possible exception of a set of values which has capacity 0, is taken by f in each neighborhood of $e^{i\theta}$; if w is an exceptional value, it is an asymptotic value either at $e^{i\theta}$ or at points $e^{i\phi}$ arbitrarily near to $e^{i\theta}$. If f is regular and bounded in D and has a radial limit of modulus 1, except on a set of capacity 0, and if f is neither a constant nor a finite Blaschke product, then f takes each value in $|w|<1$ infinitely often, with at most one exception. If f is meromorphic in D, and E is a set of capacity 0 on $|z|=1$, then each value in $C(f, e^{i\theta})-C_{R\text{-}E}(f, e^{i\theta})$ is taken in every neighborhood of $e^{i\theta}$, with at most finitely many exceptions. *G. Piranian* (Ann Arbor, Mich.)

6436:

Sakai, Akira. On minimal slit domains. Proc. Japan Acad. **35** (1959), 128–133.

Consider a plane region Ω bounded by two analytic Jordan curves γ_1, γ_2 and a closed set δ that does not meet $\gamma_1\cup\gamma_2$. Let F be the family of univalent functions $w=f(z)$ that map γ_1 and γ_2 onto circles $|w|=1$ and $|w|=r(f)<1$, with $f(z_0)=1$ for a fixed $z_0\in\gamma_1$. Denote by $A(f)$ the logarithmic area of the "image" of δ. The author establishes the existence of a unique function f_0 that minimizes the functional $2\pi\log r(f)-A(f)$ in F. The image under f_0 of δ is a set of radial slits with vanishing area.

The following unicity problem now arises: When is a given region Ω, bounded by circles $\gamma_1: |z|=1$, $\gamma_2: |z|=$ const. <1, and a closed set of radial slits, "minimal" in the sense that $f_0(z)\equiv z$. The author establishes several results: (1) A necessary and sufficient condition is that the extremal length of the family of rectifiable curves in Ω joining γ_1 to γ_2 be $(1/2\pi)\log(1/r)$; (2) if the angular measure M of the projection of δ on γ_1 vanishes, then Ω is minimal; (3) there are minimal regions Ω for which $M>0$; (4) there are non-minimal regions Ω for which the area of the slits vanishes.

Related results were independently found by E. Reich and S. Warschawsky [Pacific J. Math. **10** (1960), 965–985] and P. Nickel [Thesis, Univ. of California, Los Angeles, 1959]. *L. Sario* (Los Angeles, Calif.)

6437:

Lewandowski, Z. Sur certaines classes de fonctions univalentes introduites par P. Montel et W. Rogosinski. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 261–265. (Russian summary, unbound insert)

Let Σ_0 denote the class of function $f(z)$ univalent in $|z|<1$, and satisfying $f(0)=0$. Let S_0 be the subclass of $f(z)\in\Sigma_0$ satisfying $f(z_0)=z_0$ for some preassigned z_0, and let S_p be the subclass of Σ_0, satisfying $f^{(p)}(z_0)=1$. The author proves that if $w=f(z)\in S_0$ then the image of $|z|<1$ by $w=f(z)$ covers the circle $|w|<\frac{1}{4}(1-|z_0|^2)$, and if $f(z)\in S_p$, then the image of $|z|<1$ by $w=f(z)$ covers the circle

$$|w| < \frac{(1-|z_0|)^{p+2}}{4p!(p+|z_0|)},$$

provided Bieberbach's conjecture holds for $n=1, 2, \cdots, p$.

These results are sharp and equality holds for the Koebe functions. The author's proof follows at once from the $\frac{1}{4}$-theorem and the sharp inequality

$$|f^{(p)}(z)| < p|f'(0)|\,\frac{p+|z|}{(1-|z|)^{p+2}}$$

due to Marty [C. R. Acad. Sci. Paris **194** (1932), 1308–1310]. *W. K. Hayman* (London)

6438:

Clunie, J. On schlicht functions. Ann. of Math. (2) **69** (1959), 511–519.

The author gives two interesting examples of univalent functions. The first example, univalent in $|z|>1$, is

$$g(z) = \int_{\zeta_0}^{z} \varphi(\zeta^{-1})d\zeta = z + \sum_{n=0}^{\infty} b_n z^{-n},$$

where $\varphi(\zeta)=\exp(\lambda\sum_{n=1}^{\infty}\zeta^{k^n})$, $|\zeta|<1$, $0<\lambda<\pi/2$, k is a sufficiently large positive integer depending upon λ, and ζ_0 is fixed with $|\zeta_0|>1$. If $n=-1+k+k^2+\cdots+k^r$, then

$$|b_n| > 2^{-(\log\lambda)/\log k}n^{-1+(\log\lambda/\log k)}.$$

This shows that $O(n^{-1})$ is not the correct order for b_n. The second example, univalent and bounded in $|z|<1$, is $\Omega(z)=\int_0^z \varphi(\zeta)d\zeta$ where φ is as above with $\lambda>0$, $k>3$. One has

$$\int_{-\pi}^{\pi} \log^{+}|\Omega'(\rho e^{i\theta})|d\theta > K\{-\log(1-\rho)^{1/2}\}, \quad 0<\rho<1,$$

in which K is a positive constant. This shows that the result of Flett [Quart. J. Math. (2) **6** (1955), 59–72; MR **16**, 916] that, for any $\sigma(z)$ univalent in $|z|<1$,

$$\int_{-\pi}^{\pi} |\log^{+}\sigma'(\rho e^{i\theta})|d\theta = O(-\log(1-\rho)^{1/2})$$

is best possible. *G. Springer* (Lawrence, Kans.)

6439:

Uluçay, Cengiz. On the coefficients of schlicht functions. J. Reine Angew. Math. **202** (1959), 1–5.

Let $s(z)=z+\sum_{n=2}^{\infty}s_n z^n$ be regular and univalent in $|z|<1$, and set

$$I(r,s) = (2\pi)^{-1}\int_0^{2\pi} |s(re^{i\theta})|d\theta.$$

The author asserts that $|I(r,s)|\leq r/(1-r^2)$, with equality if and only if $s(z)=z/(1-z)^2$. However, the reviewer is not able to follow the proof.

From this bound on $I(r,s)$ it follows that for these normalized univalent functions, $|s_n|<ne/(2-1/n)$.

A. W. Goodman (Lexington, Ky.)

6440:

Uluçay, Cengiz. On inequalities in the theory of schlicht functions. J. Reine Angew. Math. **202** (1959), 6–8.

Let M be the class of functions $\mu(z)=z+\sum_{n=1}^{\infty}\mu_n z^{2n+1}$ univalent and regular in $|z|<1$. Let Ψ be the class of functions of the form $zs'(z)/s(z)=1+\sum_{n=1}^{\infty}b_n z^n$ where $s(z)$ is regular and univalent in $|z|<1$. Using the methods of the paper reviewed above the author proves that, for $n\geq 1$, $|\mu_n|<5e/9$ and $|b_n|<4e/3$. The proofs are open to the same objection as in the previous paper.

A. W. Goodman (Lexington, Ky.)

6441:

Sunyer Balaguer, F. Expansion of a function in a series of the primitives of an entire function. Mem. Acad. Ci. Madrid **5** (1959), no. 2, 61 pp. (Spanish)

The author investigates expansions of analytic functions in series of iterated integrals of a given entire function. Previous investigations were carried out by Gontcharoff [*Détermination des fonctions entières par interpolation*, Actual. Sci. Ind., no. 465, Hermann, Paris, 1937] when the entire function is a constant, so that the terms of the expansion are polynomials; by Valiron [Bull. Sci. Math. (2) **58** (1934), 22–40] for the case when one limit of integration is always the same; and briefly by Rey Pastor [C. R. Acad. Sci. Paris **200** (1935), 622–624] in the general case. The author carries the subject much further and shows that the series mimics the Taylor series with remarkable closeness. His basic hypothesis is that $\sum|z_{n+1}-z_n|$ converges, so that $z_n\to Z$, and that $|Z-z_n|<R$; let $\varphi(z)$ be an entire function with $\varphi(Z)=1$ and $\varphi(z_n)\neq 0$, and let $\varphi_n(z)$ be defined by $\varphi_n^{(n)}(z)=n!\varphi(z)$, $\varphi_n^{(k)}(z_k)=0$ $(k=0, 1, \cdots, n-1)$. Then series $\sum c_n\varphi_n(z)$ have essentially the same convergence properties as power series in $(z-Z)^n$. Previous work had obtained convergence in a circle; the author, by establishing a lower bound for the moduli of the functions $\varphi_n(z)$, can show that a series in the $\varphi_n(z)$ cannot converge in a set of appreciable size outside the circle of convergence. The author extends Jentzsch's theorem on zeros of partial sums, Ostrowski's overconvergence theorems, and the Pólya-Hurwitz theorem on changing the signs of the terms to make the circle of convergence a natural boundary. He then shows that, in many of Valiron's results which compare $\sum c_n\varphi_n(z)$ and $\sum c_n z^n$, Valiron's hypothesis $z_n=0$ can be replaced by the hypothesis that $\sum_{k=n}^{\infty}|z_{k+1}-z_k|\to 0$ sufficiently fast. In addition, under the same hypothesis, if one of these series represents an entire function of a specified rate of growth, so does the other, provided that the order exceeds both 1 and the order of $\varphi(z)$. More refined results under stricter hypotheses allow the author to carry over to the general case a number of results on the character of entire functions defined by lacunary power series.

R. P. Boas, Jr. (Evanston, Ill.)

6442:

Oberhettinger, F. On a modification of Watson's lemma. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 15–17.

The author generalizes a result of van der Waerden [Appl. Sci. Res. B **2** (1951), 33–45; MR **12**, 808] which evaluates asymptotically complex integrals of the form $\int_C P(u)e^{-\lambda u}du$, wherein λ is real. A special difficulty treated by van der Waerden is the case when a simple pole of $P(u)$ lies near a branch point of order 2 of $P(u)$. The present author considers asymptotic expansion of integrals of the form

$$f(z) = \int_0^{\infty} F(t)e^{-tz}dt$$

for the case when $F(t)$ is of the form $t^{\alpha}g(t)$, $\operatorname{Re}\alpha>-1$, and $g(t)$ is analytic around $t=0$ but has a pole of order m at $t=t_0$. Thus the branch point is of order α and the pole of order m. The expansion of the integral consists of a finite number of terms involving Whittaker functions and an asymptotic series in inverse powers of z.

M. Kline (New York, N.Y.)

6443a:

Walsh, J. L. Approximation on a line segment by bounded analytic functions: Problem β. Proc. Amer. Math. Soc. **10** (1959), 270–272.

6443b:

Walsh, J. L. Approximation by bounded analytic functions: General configurations. Proc. Amer. Math. Soc. **10** (1959), 280–285.

Ces deux articles (a et b) sont relatifs au problème suivant ("problème β") déjà étudié par l'auteur [Trans. Amer. Math. Soc. **87** (1958), 467–484; MR **20** #3298]: soit E un compact du plan, D un ouvert contenant E (notation de a); soit $f(z)$ une fonction analytique sur E; étudier, en relation avec les propriétés de $f(z)$ sur certains voisinages de E, la meilleure approximation de f sur E par des fonctions analytiques dans D et y admettant des majorations données. Les énoncés sont très voisins de ceux de l'article cité, mais relatifs à des cas non traités jusqu'alors: dans a, E est un segment de droite; dans b, E peut admettre des points intérieurs, et $D-E$ n'est pas nécessairement connexe. *J. P. Kahane* (Montpellier)

6444:

Dzyadyk, V. K. Inverse theorems on approximation of functions in domains with angular singularities. Dokl. Akad. Nauk SSSR **127** (1959), 505–508. (Russian)

Let K be a bounded closed set of the complex plane with a connected complement G. Let the boundary C_1 of K consist of finitely many arcs with continuous curvature which meet at points z_i under angles $\pi\alpha_i$, $0 \leq \alpha_i < 2$. Let further C_R ($R \geq 1$) denote the curves $|\varphi(z)| = R$, where the function $w = \varphi(z)$ is the mapping of G onto $|w| \geq 1$ for which $\lim_{z\to\infty}(z^{-1}\varphi(z)) > 0$. Let $\rho_R(z)$ denote the distance from $z \in C_1$ to C_R. Then, up to a factor $\eta(z, R)$ contained between two positive constants, $\rho_R(z)$ is equal to

$$(R-1)[|z-z_i| + (R-1)^{2-\alpha_i}]^{(1-\alpha_i)/(2-\alpha_i)},$$

where z_i is the point z_j closest to z. (The following theorems are stated by the author under certain less restrictive conditions than those given above.) For a positive continuous function $A(z)$ on C_1, the inequality $|P_n(z)| \leq A(z)$, $z \in C_1$, for a polynomial P_n of degree n implies $|P_n^{(k)}(z)| \leq MA(z)\rho_{1+1/n}(z)^{-k}$. If $f(z)$ is defined on C_1 and for each n there is a polynomial P_n of degree n such that, for $z \in C_1$, $|f(z) - P_n(z)| \leq M\rho_{1+1/n}(z)^{r+\alpha}$, where $0 < \alpha < 1$ and $r \geq 0$ is an integer, then $f(z)$ has an analytic extension into the interior of K and has a derivative $f^{(r)}(z)$ which belongs to the class Lip α. *G. G. Lorentz* (Syracuse, N.Y.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 6370, 6371, 6535.

6445:

Lelong, Pierre. Sur une classe de singularités impropres. Arch. Math. **9** (1958), 161–166.

In der vorliegenden Arbeit wird zunächst eine Klasse L_n von Teilmengen eines Gebietes D des n-dimensionalen komplexen Zahlenraumes C^n definiert. Sodann wird der Begriff invariant gegenüber biholomorphen Abbildungen gemacht und dadurch eine Definition einer Klasse von Teilmengen Λ_n einer n-dimensionalen komplexen Mannigfaltigkeit W^n gewonnen. Die Mengen der Klasse Λ_n haben interessante Eigenschaften. Jede mindestens 2-codimensionale analytische Menge gehört zu Λ_n, ebenso jede abzählbare Vereinigung von solchen Mengen. Die komplexe Mannigfaltigkeit W^n wird durch Mengen, die zu Λ_n gehören, lokal nicht zerlegt. Durch Hohlräume von solchen Mengen gehen einfach zusammenhängende Teilgebiete von W^n wieder in einfach zusammenhängende Gebiete über. Außerdem lassen sich stets plurisubharmonische Funktionen, die in einer Umgebung einer Menge aus Λ_n definiert sind, in diese Menge plurisubharmonisch durchsetzen. *H. Grauert* (Göttingen)

6446:

Gunning, R. C. On Vitali's theorem for complex spaces with singularities. J. Math. Mech. **8** (1959), 133–141.

Der Verfasser führt als "local semi-analytic space" mit Hilfe der lokalen Verzweigungsordnungen verallgemeinerte analytisch-verzweigte Überlagerungen ein, auf denen sich holomorphe Funktionen und Abbildungen definieren lassen; diese Begriffe stimmen mit den gewöhnlichen überein, falls man wieder analytisch-verzweigte Überlagerungen (local analytic spaces) zugrunde legt. Unter den angegebenen allgemeinen Bedingungen gelten das Maximumsprinzip und der folgende Satz: Jede beschränkte Folge holomorpher Funktionen enthält eine Teilfolge, die kompakt gegen eine holomorphe Funktion konvergiert; dieser Satz gilt m.m. auch für holomorphe Abbildungen endlich-blättriger Überlagerungen. Damit sind u.a. Beweise von S. Bochner und W. T. Martin [siehe J. Indian Math. Soc. (N.S.) **16** (1952), 137–146; Ann. of Math. (2) **57** (1953), 490–516; Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 726–732; MR **14**, 972] auf möglichst geringe Benutzung analytischer Hilfsmittel reduziert. Ferner ergibt sich als Anwendung: Die Gruppe der biholomorphen Automorphismen eines kompakten komplexen Raumes ist eine (komplexe) Liesche gruppe. *G. Scheja* (Münster)

SPECIAL FUNCTIONS

See also 6409, 6454, 6510.

6447:

★Rainville, Earl D. Special functions. The Macmillan Co., New York, 1960. xii + 365 pp. $11.75.

A book on special functions can be judged on its mathematical elegance, or on the usefulness of its results in day-to-day applied mathematics. On the first criterion this is a good book, though, as the author admits, it has an unusual amount devoted to generating functions.

In applying the second criterion the reviewer listed the results in the field of special functions that he has had to look up recently, and searched to see if they were included. On this criterion the book is a total failure. Thus the book can be said to cover the elements of special function theory plus a number of items that are probably useful to those doing further research on the mathematical aspects of the field. Since it is written to be read "not only by professional mathematicians, physicists, engineers and chemists, but also by well-trained graduate students in those and closely allied fields" it tries to do too much and fails to meet many practical needs. It would appear that an examination of what is actually used in this vast field should be made before writing a book that must, of necessity, select a small fragment to discuss and exclude all the rest.

The author is a master of his field and organizes the standard material quite well. The book, while at times a bit fragmentary, makes interesting reading. No serious errors were noted. *R. Hamming* (Murray Hill, N.J.)

6448:

Collins, W. D. Some integrals involving Legendre functions. Proc. Edinburgh Math. Soc. **11** (1958/59), 161–165.

Associated Legendre functions of the 1st and 2nd kinds are expressed as integrals involving corresponding functions of the same degree but of different order. For example, if $\Re\mu > -1$, $\Re\sigma > 0$ and $|1-x| < 2$, then

$$\Gamma(\sigma)P_\nu^{-(\mu+\sigma)}(x) = (x^2-1)^{-(\mu+\sigma)/2}\int_1^x P_\nu^{-\mu}(t)(x-t)^{\sigma-2}(t^2-1)^{\mu/2}dt.$$

The derivations involve elementary substitutions and interchange of the operations of integration and summation. *N. D. Kazarinoff* (Madison, Wis.)

6449:

Bailey, W. N. On the sum of a particular bilateral hypergeometric series ${}_3H_3$. Quart. J. Math. Oxford Ser. (2) **10** (1959), 92–94.

The author proves the following result:

$$\sum_{-\infty}^{\infty}\frac{(a)_n(b)_n(f+1)_n}{(d)_n(e)_n(f)_n} = \lambda\frac{\Gamma(d)\Gamma(e)\Gamma(1-a)\Gamma(1-b)\Gamma(d+e-a-b-2)}{\Gamma(d-a)\Gamma(d-b)\Gamma(e-a)\Gamma(e-b)},$$

where

$$\lambda = \{(f-a)(f-b)-(1+f-d)(1+f-e)\}/f.$$

This relation generalizes former results of Dougall [Proc. Edinburgh Math. Soc. **25** (1907), 114–132] and of Shukla [Canad. J. Math. **10** (1958), 195–201; MR **20** #1801]. Also a further generalization is given.

H. A. Lauwerier (Amsterdam)

6450:

Brafman, F. Generating functions and associated Legendre polynomials. Quart. J. Math. Oxford Ser. (2) **10** (1959), 156–160.

Consider the sequence of functions $f_n(x)$ defined by

$$f_n(x) = \frac{1}{n!}D_x^n\{(ax+b)^nF(x)\},$$

where $F(x)$ is independent of n. It is shown that if a generating function $\sum_{n=0}^{\infty}a_nt^nf_n(x)$ is known for either

$$a_n = {}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \cdots, \alpha_p;\\ \beta_1, \cdots, \beta_q\end{matrix}\ v\right]$$

or

$$a_n = \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n},$$

it is known for the other. The relation between the two generating functions is given by

$$\sum_{n=0}^{\infty}t^nf_n(x)\,{}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \cdots, \alpha_p;\\ \beta_1, \cdots, \beta_q\end{matrix}\ v\right] = (1-at)^{-1}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\left(\frac{-vt}{1-at}\right)^n f\left(\frac{x+bt}{1-at}\right).$$

It is also shown that

$$\sum_{n=0}^{\infty}t^nf_n(x)\,{}_{p+2}F_q\left[\begin{matrix}-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n, \alpha_1, \cdots, \alpha_p;\\ \beta_1, \cdots, \beta_q\end{matrix}\ v\right] = (1-at)^{-1}\sum_{n=0}^{\infty}\frac{(\tfrac{1}{2})_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\left\{\frac{vt^2}{(1-at)^2}\right\}^n f_{2n}\left(\frac{x+bt}{1-at}\right).$$

The formulas are applied to the special Jacobi polynomials $P_n^{(\alpha,-\alpha)}(x)$. *L. Carlitz* (Durham, N.C.)

6451:

Carlitz, Leonard. A generating function for the product of two ultraspherical polynomials. Boll. Un. Mat. Ital. (3) **14** (1959), 6–9. (Italian summary)

Let $C_n^\nu(x)$ denote Gegenbauer's ultraspherical polynomial as defined by

$$(1-2xz+z^2)^{-\nu} = \sum_{n=0}^{\infty}z^nC_n^\nu(x).$$

Then, as the author shows,

$$\sum_{n=0}^{\infty}\frac{n!}{(2\nu)_n}z^nC_n^\nu(\cos\alpha)C_n^\nu(\cos\beta) = \{1-2z\cos(\alpha+\beta)+z^2\}^{-\nu} \times F\left[\nu, \nu; 2\nu; \frac{4z\sin\alpha\sin\beta}{1-2z\cos(\alpha+\beta)+z^2}\right] \quad (|z|<1).$$

If $\nu = \frac{1}{2}$ the result is known [L. C. Maximon, Norske Vid. Selsk. Forh. Trondheim **29** (1956), 82–86; MR **18**, 801]. In addition, the author gives several formulae for the individual terms of the series at the left.

C. J. Bouwkamp (Eindhoven)

ORDINARY DIFFERENTIAL EQUATIONS

See also 6521, 6730, 6731, 7078.

6452:

Devianin, E. A. On the properties of equations of the first approximation in the method of averaging. J. Appl. Math. Mech. **22** (1958), 1010–1019 (713–719 Prikl. Mat. Meh.).

The author considers a system of total differential equations linear with constant coefficients with the exception of one non-linear term involving but one of the unknown functions and occurring in only one of the equations. This system, under certain assumptions on the coefficients and relative size of the coordinates, is transformed to normal coordinates and averaged over the coordinates. The resulting system then takes on the simple form $dx_i/dt = \partial\Psi/\partial x_i$. Application is made to the motion about the equilibrium of the gyroscopic pendulum, assuming a small amount of friction along one of the axes.

R. G. Langebartel (Urbana, Ill.)

6453:

Kalaba, Robert. On nonlinear differential equations, the maximum operation, and monotone convergence. J. Math. Mech. **8** (1959), 519–574.

In der Anfangswertaufgabe $u' = f(u, x)$, $u(0) = c$ sei $f(u, x)$ eine streng konvexe Funktion: $f(u) = \mathrm{Max}_v[f(v) + (u-v)f'(v)]$ (Abkürzung: $f(u) = f(u, x)$, $f'(u) = \partial f/\partial u$). Dann ist die Lösung $u = \mathrm{Max}_v w[x, v]$, wobei w der linearen Differentialgleichung $w' = f(v) + (w-v)f'(v)$ und $w(0) = c$ genügt. Diese Beziehung wurde für die Riccatische Differentialgleichung von Bellman [Proc. Nat. Acad. U.S.A. **41** (1955), 743–746; MR **17**, 632] benutzt. Für die

Lösung u gilt $u \geqq w[x, v]$ und die "Newtonsche" Näherungsfolge $u_{n+1} = w[x, u_n]$ konvergiert monoton gegen u. Entsprechende Ergebnisse werden jeweils ausführlich hergeleitet für die erste Randwertaufgabe bei den Differentialgleichungen $u'' = f(u, x)$, $u'' = f(u', u, x)$, $u_{xx} + u_{yy} = f(u, x, y)$ und das Problem $u_t - u_{xx} = f(u, x, t)$, $u(0, t) = u(1, t) = u(x, 0) = 0$. Die Überlegungen lassen sich auch auf andere Aufgaben $L[u] = f(u, x)$ $(x \in R^n)$ mit einem linearen Operator L übertragen, für den (n.a.) aus $Lz - f'(v)z \geqq 0$ die Beziehung $z \geqq 0$ folgt. Für die genannten gewöhnlichen Differentialgleichungen rechnet der Verf. numerische Beispiele. *J. Schröder* (Hamburg)

6454:

Erugin, N. P. On the theory of Painlevé's first equation. Dokl. Akad. Nauk BSSR **2** (1958), 3–6. (Russian)

The first transcendent of Painlevé satisfies $W'' = 6W^2 + z$ and may be expressed as $W = (\zeta'^2 - \zeta\zeta'')/\zeta^2$, where ζ is an entire function of z [E. L. Ince, *Ordinary differential equations*, Dover Publications, New York, 1944; MR **6**, 65; p. 351]. The author derives the equation

$$(*) \qquad \zeta\zeta^{(iv)} - 4\zeta'\zeta''' + 3\zeta''^2 + z\zeta^2 = 0$$

for the function ζ and determines the values of ζ', ζ'', ζ''' at $z = 0$ with $\zeta(0) = 1$ which correspond to the initial conditions $W(0) = W_0$, $W'(0) = W_1$. In addition he studies the function ζ when W is prescribed to have a pole at z_0 instead of given conditions at $z = 0$. The solutions of (*) which are zero at z_0 are multiples of $\zeta = \sum_1^\infty \alpha_n t^n$ where $t = z - z_0$, $\alpha_1 = 1$ and the α_n for $n \geqq 2$ are functions of the two parameters α_2 and α_7. From the known expansion of W in the vicinity of a pole [Ince, p. 346] the author finds $\alpha_7 = \alpha_2^6/720 + z_0\alpha_2^2/240 - \alpha_2/420$ with α_2 arbitrary. {This seems to be in error and the reviewer believes that the linear term in this polynomial should be $3\alpha_2/280$.} Finally the author suggests an approach to the problem of approximating the function W in the vicinity of the pole z_0 nearest to the origin when W_0 and W_1 are prescribed, which represents an alternative to that of H. T. Davis [(1) of Davis, Scott, Springer and Resch, *Studies in differential equations*, Northwestern Univ. Press, Evanston, Ill., 1956; MR **18**, 653]. *C. E. Langenhop* (Ames, Iowa)

6455:

Adem, Julián. Matrix differential systems with a parameter in the boundary conditions and related vibration problems. Quart. Appl. Math. **17** (1959), 165–171.

The problem of a vibrating string with attached point masses is formulated as a self-adjoint eigenvalue problem for a system of ordinary differential equations with the eigenvalue parameter appearing in the boundary conditions as well as the differential equation. It is shown that such a problem has the usual properties of a self-adjoint eigenvalue problem.

H. F. Weinberger (College Park, Md.)

6456:

Pozharitskii, G. K. On a certain property of the characteristic numbers of the solutions of differential equations. J. Appl. Math. Mech. **22** (1958), 999–1004 (707–710 Prikl. Mat. Meh.).

The author modifies a result of a previous note [Prikl. Mat. Meh. **19** (1955), 481–484; MR **17**, 366] concerned with the connection between the (Lyapunov) characteristic numbers of the solutions of (1) $\dot{x} = P(t)x$ and those of (2) $\dot{x} = P(t)x + X(x, t)$ with X holomorphic in x and $O(\|x\|^2)$ for small $\|x\|$ uniformly in $t \geqq 0$. Under the assumptions that (1) is regular and that the coefficients $P_s^{(m_1 \cdots m_n)}$ in the expansion of X_s satisfy $|P_s^{(m_1 \cdots m_n)}| < MA^{-\sum m_i}$ $(A, M > 0)$ for $t \geqq 0$, he proves that if $\lambda_1 \geqq \cdots \geqq \lambda_k > 0$ are characteristic numbers of (1) then (2) has k solutions with the same characteristic numbers. Moreover, the first p components of the initial point of a solution of (2) with characteristic number $\lambda_1 > \lambda_p$ can be chosen arbitrarily, the remaining components being holomorphic functions of them.

H. A. Antosiewicz (Los Angeles, Calif.)

6457:

Polniakowski, Z. Polynomial Hausdorff transformations. II. Regularity theorems and asymptotic properties of solutions of linear difference and differential equations. Ann. Polon. Math. **6** (1959), 111–133.

[For part I, see same Ann. **5** (1958), 1–24; MR **20** #7170.] The author studies boundedness and more general asymptotic behavior of solutions of linear difference and differential equations

$$Ly \equiv y + \sum_{\nu=1}^{k} \lambda_\nu \nabla^\nu y = f, \quad Ly \equiv y + \sum_{\nu=1}^{k} \lambda_\nu y^{(\nu)} = f.$$

The principal results are stated in terms of Wronskians W formed with linearly independent solutions $\varphi_1, \varphi_2, \cdots$ of the homogeneous equations $Ly = 0$. The Wronskians enter through their ratios

$$w_\nu = W(1, \varphi_1, \cdots, \varphi_{\nu-1})/W(\varphi_1, \varphi_2, \cdots, \varphi_\nu) \qquad (\nu = 2, \cdots, k;\ w_1 = 1/\varphi_1).$$

The following theorem for differential equations on an interval (a, b) is typical. Suppose that (i) $w_\nu(x) \to 0$ as $x \to x_0$ and $|\int_x^{x_0} |w_\nu'|| \leqq C|w_\nu(x)|$ on some neighborhood I of x_0; (ii) $w_\nu(x) \neq 0$ on (a, b); (iii) the products $w_\nu' f$ are continuous on (a, b) and $\lim \sup_{x \to x_0} |f(x)| \leqq M$. Then the differential equation $Ly = f$ has a solution $\bar{y}$ on (a, b) such that $\lim \sup_{x \to x_0} |\bar{y}(x)| \leqq MC^k$. If in the above hypotheses (i) is replaced by (i') $|w_\nu(x)| \to \infty$ as $x \to x_0$ and $|\int_y^x |w_\nu'|| \leqq C|w_\nu(x)|$ for all $x, y \in I$, then all solutions of the differential equation $Ly = f$ satisfy the inequality given for $\bar{y}(x)$. In special cases the results are refined to asymptotic equalities. As an application the author proves Tauberian theorems for Hausdorff transforms.

J. Korevaar (Madison, Wis.)

6458:

Starzhinskii, V. M. On the stability of the trivial solution of a linear system with periodic coefficients. J. Appl. Math. Mech. **22** (1958), 907–921 (646–656 Prikl. Mat. Meh.).

Let $\dot{x} = A(t)x$ where $A(t + \omega) = A(t)$ and $\int_0^\omega \operatorname{trace} A(s)ds \leqq 0$. The author is concerned with determining the region of stability for the characteristic equation

$$\sum (-1)^{n-k} a_{n-k} \rho^k = 0.$$

If $A(t) = J_{2m}H(t)$ (i.e., if the system is in canonical form) and if the characteristic equation is reciprocal, he applies a theorem of Herglotz to obtain various inequalities for the coefficients a_k; for example, if $a_2 < -m + 1 - (-1)^m$ then the system is unstable whatever the remaining a_k.

In the general case with $0<a_n<1$ the conditions $|a_k|<\binom{n}{k}$ are necessary for stability, and either $a_1<-n+1-(-1)^n a_n$ or $a_1>n-(1-a_n)$ is sufficient for instability. The author also considers systems $\ddot{y}+P(t)y=0$ with $P(t)$ piece-wise continuous and periodic.

H. A. Antosiewicz (Los Angeles, Calif.)

6459:

Langenhop, C. E. Note on almost periodic solutions of non-linear differential equations. J. Math. Phys. **38** (1959/60), 126–129.

Let x be an n-dimensional complex vector, A a constant matrix without purely imaginary characteristic roots, $b(t)$ an almost periodic function, λ a real constant, k and ω positive constants, and $f(t, x)$ a vector function which is almost periodic for fixed x and satisfies the conditions $f(t, 0)=0$ and $(x-y)^{-1}(f(t, x)-f(t, y))\to 0$ uniformly in t when $(x, y)\to(0, 0)$. Let, further, $g(t, y)$ be a continuously differentiable vector function periodic in t, such that $\dot{y}=g(t, y)$ has a periodic solution $h(t)$ and no characteristic exponents of $v=(\partial g_i(t, h(t))/\partial y_j)v$ have a zero real part. The author proves that each of the differential equations $\dot{x}=Ax+f(\lambda t, x)+kb(\omega t)$ and $\dot{y}=g(t, y)+kb(\omega t)$ has an almost periodic solution when k is small enough. If $b(t)$ has a bounded indefinite integral, it is enough to require that $k(1+\omega)^{-1}$ is small enough. *H. Tornehave* (Copenhagen)

6460:

Gel'fand, I. M.; and Pyateckiĭ-Šapiro, I. I. A theorem of Poincaré. Dokl. Akad. Nauk SSSR **127** (1959), 490–493. (Russian)

On a compact differentiable manifold M let there be given a flow defined by a vector field. Suppose $t_1<t_2<\cdots$ goes to infinity. Let $p\in M$ and suppose V is a simply connected neighborhood of p such that pt_i belongs to V for all i. If we complete the orbit from p to pt_i by an arc lying entirely in V the homology class A_i of the resulting closed curve is independent of the arc we use to close up the orbit. We can ask about the asymptotic behavior of A_i and in particular about $\lim_{i\to\infty} A_i/t_i$. On the torus this leads to the rotation numbers of Poincaré. The authors' first result states that if the flow is metrically transitive $\lim_{i\to\infty} A_i/t_i$ exists and has the same value for almost all points p. As stated in a note added in proof, the ideas up to this point essentially constitute an independent derivation of a part of a recent paper of the reviewer [Ann. of Math. (2) **66** (1957), 270–284; MR **19**, 568].

Next let G be the group of real 2×2 unimodular matrices and let Γ be a discrete subgroup of G containing no elements of finite order. The authors consider in turn the flows defined on G/Γ by each of the following one-parameter subgroups of G:

$$\text{(a)}\quad \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad \text{(b)}\quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

In case (a), $A_t=O(t_t^{1/2+\varepsilon})$ for any positive ε and almost all points p. In case (b), $A_t=O(\log t)$ for all p.

S. Schwartzman (Baltimore, Md.)

6461:

Lee, Wen-yung. The topological structure of the distribution of the integral curves with one singular point on the torus. Acta Math. Sinica **9** (1959), 181–190. (Chinese. English summary)

"In this paper we consider the topological structure of the distribution of the integral curves with one singular point on the torus. First of all, we define three kinds of sectors. We consider the neighborhood of the singular point by separating it into sectors. Using the property of the torus and the fact that the sum of the index of the singular point is equal to zero, we obtain three classes of fundamental figures. We define five classes of operations. We prove that from the three classes of fundamental figures and the five classes of operations we can obtain all possible topological structure with one singular point on the torus. Finally, we prove the differentiability of the figure and consider also the case of a finite number of singular points on the torus." (Author's summary)

C. T. Yang (Philadelphia, Pa.)

6462:

Kuzmak, G. E. On the computation of asymptotic solutions corresponding to nonclosed integral curves of the "standard" equation. Dokl. Akad. Nauk SSSR **125** (1959), 992–995 (Russian); translated as Soviet Physics. Dokl. **4**, 309–312.

The author investigates the equation

$$(*)\qquad y''+\varepsilon f(\varepsilon t, y)y'+F(\varepsilon t, y)=0,$$

where $\varepsilon>0$ is a small parameter. The "standard" equation referred to in the title is

$$(**)\qquad \phi^2(\varepsilon t)(\partial^2 y_0/\partial\omega^2)+F(\varepsilon t, y_0)=0,$$

where ϕ is determined by a complicated expression and ω is the integral of ϕ. In an earlier paper [Dokl. Akad. Nauk SSSR **120** (1958), 461–464; MR **21** #169] he considered the oscillatory case. Here he extends his method to the case where the integral curves in the phase plane y_0, $\partial y_0/\partial\omega$ are non-closed. The author shows that by an appropriate choice of the arbitrary functions that enter into the solution of (**) one may obtain functions y_0 and y_0' which are close (within terms of order ε^2) to the solution of (*) and its derivative respectively.

K. S. Miller (New York, N.Y.)

6463:

Grabar', M. I. Isomorphism of dynamical systems differing only in time. Dokl. Akad. Nauk SSSR **126** (1959), 931–934. (Russian)

In a previous paper [same Dokl. **109** (1956), 431–433; MR **19**, 569] the author discussed the case of two dynamical systems defined on a compact space and having identical trajectories but different time variables. He found sufficient conditions for the two systems to be unitarily isomorphic. In the present paper, further sufficient conditions are found which imply the previous ones.

G. Hufford (Seattle, Wash.)

6464:

Bellman, Richard; and Cooke, Kenneth L. Stability theory and adjoint operators for linear differential-difference equations. Trans. Amer. Math. Soc. **92** (1959), 470–500.

Using methods standard in differential equations theory the authors derive stability theorems for certain difference-differential equations of mixed difference type, after illustrating these methods with differential equations. In

general such theorems are based upon integral representations of the solution, as, for instance, those obtained for differential equations by the method of variation of parameters. The representations in this paper are derived in the usual way from Green's theorems so that the behavior of a solution is determined in terms of that of an adjoint difference-differential equation. Boundedness, as the independent variable tends to infinity, is established for certain systems of differential equations and difference-differential equations under various conditions, notably the assumed boundedness of some other equations having coefficients to which those of the original equations tend asymptotically. *E. Pinney* (Berkeley, Calif.)

6465:

Foiaş, C.; Gussi, G.; et Poenaru, V. Sur les solutions généralisées de certaines équations linéaires et quasi linéaires dans l'espace de Banach. Rev. Math. Pures Appl. **3** (1958), 283–304.

The authors investigate the initial value (Cauchy) problem for $du/dt = A(t)u$ in a Banach space X. $A(t)$ is an unbounded operator satisfying the following conditions: (i) There exist orthogonal projections $\{J_k\}$ such that $I = \sum J_k$; $J_n A(t) \subseteq A(t) J_n$; $A(t)J_n$ is bounded on $X_n = J_n X$; and $A(t)J_n$ is continuous in norm as a function of t. (ii) There exists an integrable $r(t)$ such that for $\varepsilon > r(t)$, $[\varepsilon I - A(t)]^{-1}$ is bounded in norm by $[\varepsilon - r(t)]^{-1}$. The space X is imbedded in a complete and metrizable locally convex space in which the Cauchy problem is shown to be well set. If the initial value $u(t_0)$ for this new Cauchy problem is in X, then the solution $u(t)$ is also; it is a generalized solution of the original Cauchy problem in X. With an additional assumption on $A(t)$, the generalized solution is also an ordinary solution, and in this case one can construct the generalized solution of $du/dt = A(t)u + f(t, u)$.

R. W. McKelvey (Boulder, Colo.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 6465, 6539, 6540, 6702, 6703, 6706, 6887.

6466:

Merli, Luigi. Esistenza e unicità degli integrali di un'equazione alle derivate parziali della forma

$$X_1(x_1, \cdots, x_n)\frac{\partial z}{\partial x_1} + \cdots + X_n(x_1, \cdots, x_n)\frac{\partial z}{\partial x_n} = f(x_1, \cdots, x_n, z).$$

Ann. Mat. Pura Appl. (4) **46** (1958), 97–107.

An existence theorem is established for the solution of $X(x, y)z_x + Y(x, y)z_y = f(x, y, z)$ such that $z(x_0, y) = \phi(y)$, under conditions which insure that Y/X and $F = f/X$ are Lipschitzian, with respect to y and z, respectively, in a given region. The method of proof consists in showing that the operator $\mathcal{D} = X(x, y)\cdot\partial/\partial x + Y(x, y)\cdot\partial/\partial y$ can be replaced by the composition of four operators:

$$X(x, y)\cdot\begin{Bmatrix} y \\ \omega(x, y)\end{Bmatrix}\cdot\frac{\partial}{\partial x}\cdot\begin{Bmatrix} y \\ \gamma(x, y)\end{Bmatrix},$$

where $y = \gamma(x, c)$ is a general solution of the ordinary differential equation $y' = Y/X$, and where $c = \omega(x, y)$ is the inverse of γ, for a given x; the substitution operator $\begin{Bmatrix} y \\ \omega(x, y)\end{Bmatrix}$ is defined by $\begin{Bmatrix} y \\ \omega(x, y)\end{Bmatrix}u = u[x, \omega(x, y)]$. The given problem is equivalent to an integral equation:

$$z(x, y) = \varphi[\gamma(x_0, \omega(x, y)] + \int_{x_0}^{x} F[\xi, \gamma(\xi, \omega(x, y)), z(\xi, \gamma(\xi, \omega))]d\xi$$

which is shown to have a unique solution, by the method of successive approximations. This can be generalized to n independent variables, but the general statement is not given. *D. L. Bernstein* (Towson, Md.)

6467:

Pucci, Carlo. Proprietà di massimo e minimo delle soluzioni di equazioni a derivate parziali del secondo ordine di tipo ellittico e parabolico. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 370–375; **24** (1958), 3–6.

In these two papers the author proves a theorem concerning the behavior of the solution of $L(u) = 0$ (where $L(u) = \sum a_{ij}u_{ij} + \sum b_i u_i + cu$, and $\sum a_{ij}\lambda_i\lambda_j \geqq 0$), in the neighborhood of a maximum or minimum point, occurring on the frontier of the region within which the equation is satisfied. The conditions on the coefficients are shown, by examples, to be necessary. The theorem, which is a generalization of a result given by E. Hopf for elliptic equations and L. Nirenberg for parabolic equations, is essentially the following. Let u be a real function of class C^2 in S, an open sphere in R^m, with center at η and radius r_0 for which $L(u) \leqq 0$ in S and u is continuous in $\bar{S}$. Let x^0 be a point on the boundary of S where $u(x^0) \leqq 0$ such that, for all $x \in S$, $u(x) < u(x^0)$. If the coefficients satisfy certain conditions, then the lower limit of the ratio $[u(x) - u(x^0)]/\overline{xx^0}$, for $x \in$ a segment in S terminating at x^0, is positive. A second theorem concerns conditions under which u can attain a maximum or minimum in the interior of a region. From these two theorems, various corollaries are given, such as the fact that, when $c = 0$, u has no maximum or minimum in the interior of T (under appropriate hypotheses).

D. L. Bernstein (Towson, Md.)

6468:

Littman, Walter. A strong maximum principle for weakly L-subharmonic functions. J. Math. Mech. **8** (1959), 761–770.

Let

$$Lf \equiv \sum a_{ij}(x)\partial^2 f/\partial x_i \partial x_j + \sum b_i(x)\partial f/\partial x_i$$

be an elliptic operator, defined in a domain D in E^n ($a_{ij} \in C^2(D)$, $b_i \in C^1(D)$); let L^* denote the formal adjoint of L. The author says that the function $u(x)$ is "weakly L-subharmonic" in D if u is measurable and essentially bounded from above (i.e., bounded from above except for a possible set of measure zero) in D and satisfies the inequality $\int_D uL^*v \geqq 0$ for all non-negative $v(x)$ in $C^2(D)$ having compact support in D. Results: (1) Let u be weakly L-subharmonic in D and assume its essential supremum M (over D) almost everywhere in an open set in D. Then $u = M$ almost everywhere in D. (2) Let u be weakly L-subharmonic in a domain D. If u assumes its

essential supremum M (over D) at a point of continuity y of u then $u=M$ almost everywhere in a neighborhood of y. *A. Huber* (Muenchenstein)

6469:

Colautti, Maria Pia. Sulla determinazione di un sistema completo di vettori. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 268–272.

The author considers the following problem proposed by A. N. Lowan: Let A be a domain in Euclidean space with boundary Σ. Given a system w_m of functions C^2 in a domain containing $A+\Sigma$, when does the system of pairs $[\Delta_2 w_m, w_m]$ span the Hilbert space $\mathscr{S}$, the direct sum of $L^2(A)$ and $L^2(\Sigma)$? (Δ_2 is the Laplace operator.) She shows that this is the case provided A has some weak smoothness properties and the functions w_m are dense in the space of C^2 functions on some rectangular domain R containing $A+\Sigma$. In fact, suppose the pair $[\psi_1, \psi_2]$ is orthogonal to all of the $[\Delta_2 w_m, w_m]$ and let $v(y)=\int_\Sigma \psi_2(t)s(t, y)d\sigma_t$, where s is the fundamental solution for Δ_2. For y outside A, she modifies $s(\cdot, y)$ to be C^2 and shows that $v(y)=0$. For y inside A, she shows that $v(y)$ is 0 because v is a harmonic function with zero boundary data (in the weak sense). But ψ_2 may be expressed almost everywhere on Σ as a limit of differences of values of derivatives of v, so that $\psi_2=0$ a.e. on Σ. It then follows easily that $\psi_1=0$ a.e. in A. The author states a generalization to second order elliptic operators with variable coefficients. *E. Nelson* (Princeton, N.J.)

6470:

Littman, Walter. Résolution du problème de Dirichlet par la méthode des différences finies. C. R. Acad. Sci. Paris **247** (1958), 2270–2272.

The generalized Dirichlet problem, relative to Ω, an open bounded set in R^N, for a strongly elliptic operator $L=\sum(-1)^m D^\alpha a_{\alpha\beta}(x)D^\beta$, with vanishing Dirichlet data, consists in finding a $u\in\mathring{H}_m(\Omega)$ such that for all $v\in\mathring{H}_m$: $\sum(D^\alpha v a_{\alpha\beta}(x)D^\beta u)_0=(v,f)_0$. Imbedding Ω in a unit cube Q, and replacing differentiation with respect to x_i by a central difference with respect to the ith coordinate, where the mesh $h=2^{-n}$ for any integer n, the operator L is replaced by a difference operator

$$L_n=\sum(-1)^m D_n^\alpha a_{\alpha\beta}(x)D_n^\beta$$

which we can apply to functions u in G_n^{m+2}, where G_k^s is a vector space generated by the characteristic functions of the cubes of the nth subdivision of Q. Then $L_n u\sim f$ means that $(L_n^* v, u)_0=(v,f)_0$ for all $v\in G_n^{m+2}$.

It is then proved that there exists a constant $\lambda_0\geqq 0$ such that for all $f\in L^2(\Omega)$, all integers $n\geqq 1$, and all $\lambda\geqq\lambda_0$, there is a unique element u_n of G_n^{m+2} satisfying $(L_n+\lambda)u_n\sim f$. As $n\to\infty$, u_n converges in $L^2(\Omega)$ toward a $u\in\mathring{H}_m(\Omega)$ satisfying $(L+\lambda)u=f$. In addition, $D_n^\alpha u_n$ converges to $D^\alpha u$, in $L^2(\Omega)$, strongly for $|\alpha|\leqq m-1$ and weakly for $|\alpha|=m$. For all $k\geqq 0$, there is an integer r such that when $f\in H_r(\Omega)$, then $u\in C^k(\Omega)$ and the $D_n^\alpha u_n$ converge to $D^\alpha u$, uniformly in every compact subset of Ω. A second theorem says that if the only solution of $Lu=0$, for $u\in\mathring{H}_m(\Omega)$, is $u=0$, then for all $f\in L^2(\Omega)$ there exists an integer $n_0>0$ such that for all $n\geqq n_0$ there is a $u_n\in G_n^{m+2}$ which satisfies $L_n u_n\sim f$ and is unique. As $n\to\infty$, u_n converges toward $u\in\mathring{H}_m(\Omega)$ satisfying $L(u)=f$. (The differences converge to the derivatives as in the preceding theorem.) Finally in a third theorem it is shown that λ_* is an eigenvalue of L (in the sense of $\mathring{H}_m(\Omega)$), if there is a sequence $\{\lambda_i\}$, converging toward λ_*, such that, for each $i=1, 2, \cdots$, λ_i is an eigenvalue of L_{n_i} (in the sense of $G_{n_i}^{m+2}$), where $n_i\to\infty$. (This condition is also necessary if L is formally self-adjoint.) The notation and definitions throughout are those given by Nirenberg in Comm. Pure Appl. Math. **8** (1955), 648–675 [MR **17**, 742]. *D. L. Bernstein* (Towson, Md.)

6471:

Wienholtz, Ernst. Bemerkungen über elliptische Differentialoperatoren. Arch. Math. **10** (1959), 126–133.

Some properties of the elliptic operator

$$Ku=-\sum_{\mu,\nu=1}^{N}\frac{\partial}{\partial x_\nu}\left(a_{\mu\nu}(x)\frac{\partial u}{\partial x_\mu}\right)+2i\sum_{\nu=1}^{N}b_\nu(x)\frac{\partial u}{\partial x_\nu}+i\sum_{\nu=1}^{N}\frac{\partial b_\nu}{\partial x_\nu}u+q(x)u$$

with smooth, real coefficients are studied in

$$\mathfrak{D}_K=\{u(x)\,|\,u\in C^\infty\cap\mathfrak{H}, Ku\in\mathfrak{H}\},$$

where H is the set of square integrable functions in N-dimensional Euclidean space. Among other things it is proved that a non-real number λ_0 cannot be an eigenvalue of K in $\mathfrak{D}_K$ unless all other complex numbers λ satisfying $(\operatorname{Im}\lambda)(\operatorname{Im}\lambda_0)>0$ are also eigenvalues. From this it follows that K is hermitian in $\mathfrak{D}_K$ if two numbers λ_1 and λ_2 exist such that $(\operatorname{Im}\lambda_1)(\operatorname{Im}\lambda_2)<0$, neither of which is an eigenvalue of K in $\mathfrak{D}_K$. These statements generalize certain results of A. I. Martin [Quart. J. Math. Oxford Ser. (2) **5** (1954), 212–227; MR **16**, 827] for the operator $T=-\Delta+c(x)$ in two dimensions. It is also shown that if $c(x)$ is real, continuous, and satisfies $c(x)\geqq -a|x|^2-b$, then T is hermitian in $\mathfrak{D}_T$. A uniqueness theorem for Green's function is also given. In the plane, the latter results were first proved by E. C. Titchmarsh [Canad. J. Math. **1** (1949), 191–198; MR **10**, 537]. *M. Schechter* (New York, N.Y.)

6472:

Wolf, František. On the essential spectrum of partial differential boundary problems. Comm. Pure Appl. Math. **12** (1959), 211–228.

The first chapter discusses a densely defined, closed operator A in abstract Hilbert space. Among other things it is shown that A does not have a singular sequence (i.e., a sequence $\{u_n\}$ of elements in the domain $\mathfrak{D}_A$ of A having no convergent subsequence and such that $Au_n\to 0$) if and only if its range is closed and its null space finite-dimensional. For other interesting results we refer to the paper. It is pointed out that some of them are due to Gohberg and Krein [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 43–118; MR **20** #3459].

In the main part of the paper, A is chosen to be an elliptic partial differential operator in a domain $G\subseteq E^n$ with Dirichlet boundary conditions on the boundary bG. The ellipticity of A is permitted to break down at boundary points, a situation which may be accompanied by the emergence of an essential spectrum $\sigma_e(A)$ (i.e., spectral singularities which are not isolated eigenvalues of finite multiplicity). It is shown that points of $\sigma_e(A)$ have a

"local origin". Each one can be traced to a section of the boundary where ellipticity breaks down. As long as A is unchanged in the neighborhood of such a section, the specific point of the essential spectrum remains undisturbed, no matter how the domain or coefficients of A are changed elsewhere. Hence at every boundary point x a "local essential spectrum" $\sigma_e(x)$ may be introduced which is the subset of $\sigma_e(A)$ "caused" by x. It is shown that an operator which behaves like A in the neighborhood of x generates the same $\sigma_e(A)$.

Specifically, let $A=\sum_{|\mu|\leq m} a_\mu D^\mu$ have continuous coefficients in G and let S be the set of those points $x \in bG$ satisfying one of the following conditions: (1) x is isolated, (2) A is not elliptic at x, (3) some coefficient of A is not continuous at x. Points of $bG-S$ are called regular and bG is assumed of class C^m near such points. The domain $\mathfrak{D}_A$ of A is to contain only those functions which vanish on the boundary together with their first $\frac{1}{2}m-1$ derivatives in the neighborhood of regular points and is to be such that A is closed in $L^2(G)$. The abstract theorem quoted above shows that $\lambda \in \sigma_e(A)$ if and only if $I\lambda-A$ has a singular sequence. However, every singular sequence has a subsequence which converges in subdomains having positive distance from S. Thus one is led to define the local essential spectrum $\sigma_e(x)$ in the following way. For any $x \in bG$, $\lambda \in \sigma_e(x)$ means that there is a singular sequence $\{u_n\}$ for $I\lambda-A$ such that $\{u_n\}$ is singular in every neighborhood of x. It is easily shown that $\sigma_e(A)=\bigcup_{x \in bG} \sigma_e(x)$. If

$$B = \sum_{|\mu|=|\nu|=m} D^\mu a_{\mu\nu} D^\nu, \quad A = B + \sum_{|\mu|<2m} P_\mu D^\mu$$

and the $p_\mu(x)=0$ near singular points of B, it is shown that $\sigma_e(A)=\sigma_e(B)$ is on the real axis. The paper ends with several sufficient conditions for $\sigma_e(A)$ to be empty.

The proofs of these theorems employ $L^2(G)$ coerciveness inequalities for the Dirichlet problem. Since these inequalities have also been proved for more general boundary problems, the present results apply to such problems as well. *M. Schechter* (New York, N.Y.)

6473:

Wolf, František. On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 142–147.

Let E^3 be the real three-dimensional Euclidean space with points $x=(x_1, x_2, x_3)$. Let L^2 be the space of functions $f(x)$ for which $|f|^2$ is integrable over E_3. Let

$$\tilde{A} = \sum_{i=1}^{3} \left[\frac{-\partial^2}{\partial x_i^2} + a_i(x)\frac{\partial}{\partial x_i}\right] + b(x),$$

where a_i, b are in $L^2(E^3)$ and where A operates on the space of infinitely often differentiable functions. Let S be a bounded regular surface in E_3 and let $\tilde{B}$ denote the operator $\tilde{A}$ with its domain restricted to functions which vanish in some neighborhood of S. Finally, by an extension process for whose details we refer to the paper, the operators $\tilde{A}$, $\tilde{B}$ are extended to operators A, B, respectively, operating in some Hilbert space.

The main result of the paper is as follows: There exists a real λ_1 such that, for all λ in the complex plane whose real part is less than λ_1, the operator $(\lambda I-A)^{-1}-(\lambda I-B)^{-1}$ is compact.

Various consequences are derived from this main result. We mention the following two. (1) The essential spectrum of A coincides with the essential spectrum of B; here the essential spectrum of an operator C is defined as the set of points in the λ-plane which are either eigenvalues of infinite multiplicity, or for which the range of $\lambda I-C$ is not closed. (2) The following condition is sufficient for A to have a discrete spectrum: the $a_i(x)$ are real and $\lim_{|x|\to\infty}$ [inf real part of $b(x)-\sum(\partial a_i/\partial x_i)$] $=+\infty$.

E. H. Rothe (Ann Arbor, Mich.)

6474:

★Mizohata, Sigeru. Hypoellipticité des opérateurs différentiels elliptiques. La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 165–177. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

A linear elliptic partial differential equation of arbitrary order with coefficients of class C^∞ is considered. A new proof is presented of the theorem that any distribution that solves such an equation is itself infinitely differentiable. The operator involved is therefore said to be hypoelliptic. The method is based on Fourier transforms, the construction of a parametrix, and suitable function spaces. *P. R. Garabedian* (New York, N.Y.)

6475:

Mizohata, Shigeru; et Hamada, Yûsaku. Hypoellipticité. Proc. Japan Acad. **34** (1958), 482–486.

The authors give an alternate demonstration, based on the material described in the preceding review, of a result due to Malgrange about hypoelliptic differential operators.

P. R. Garabedian (New York, N.Y.)

6476:

Bergendal, Gunnar. Sur la convergence et la sommabilité des transformations de Fourier associées à un opérateur différentiel elliptique. C. R. Acad. Sci. Paris **247** (1958), 1820–1822.

This note gives generalizations, to any elliptic operator, of results on the convergence and summability of the Fourier series associated with Sturm-Liouville operators and Laplacians. For example, if $a(D)=\sum a_\alpha D^\alpha$ ($|\alpha| \leq m$) is a differential operator with constant coefficients, defined on $C_0^\infty(S)$, where S is an open set in R^n, and A is the extension of the operator a in the space $L^2(S)$, which is assumed self-adjoint and bounded below by λ_0, then the spectral family $\{E_\lambda\}$ of A has E_λ an integral operator of kernel $e(\lambda, x, y)$. Defining

$$I^k\sigma(\lambda) = \Gamma(k)^{-1}(\lambda-\lambda_0)^{1-k}\int_{\lambda_0}^{\lambda}(\lambda-\mu)^{k-1}d\sigma(\mu) \quad (k \geq 1)$$

it is shown that, as $\lambda \to +\infty$,

$$I^k\partial^{\alpha,\beta}[e(\lambda, x, y)-e_0(\lambda, x, y)] = O(1)\lambda^{(n+|\alpha|+|\beta|-k)/m}.$$

Here e_0 is the kernel corresponding to the unique extension $A=A_0$, when $S=R^n$, so that

$$e_0(\lambda, x, y) = (2\pi)^{-n}\int_{a(\zeta)<\lambda} e^{-i(x-y)\zeta}\,d\zeta.$$

Then, if $e(\lambda, f, y)=\int_S e(\lambda, x, y)f(x)dx$, it is shown that, as $\lambda \to \infty$,

$$I^k\partial^\alpha[e(\lambda, f, y)-e_0(\lambda, f, y)] = O(1)\lambda^{(n+2|\alpha|+1-2k)/2m}.$$

These estimates are uniform in every compact part of $S \times S$ and of S, respectively. These results, and others of similar character, are generalizations of theorems due to Gårding and Ganelius. *D. L. Bernstein* (Towson, Md.)

6477:

Vol'pert, A. I. The first boundary problem for elliptic systems of differential equations. Dokl. Akad. Nauk SSSR **127** (1959), 487–489. (Russian)

Consider the elliptic system

$$(1) \qquad \sum_{0 \le k+l \le n} A_{kl}(z) \frac{\partial^{k+l} u}{\partial x^k \partial x^l} = f(z), \quad z = (x, y),$$

where $A_{kl}(z)$ is a square matrix of order p defined in a closed bounded region $D+\Gamma$, and n is an even number. A column vector $u(z)$ of height p is called a solution of the homogeneous first boundary problem in D if (2) $\partial^k u/\partial y^k|_\Gamma = 0$ $(k=0, 1, \cdots, \frac{1}{2}n-1)$, where y is directed along the normal to Γ. The index of the problem (1), (2) is defined as the difference in the number of solutions of (1), (2) with $f(z)=0$ and the number of solutions of the associated transpose problem. The author gives a procedure for calculating the index in any particular case. A particular consequence is the determination of a simple sufficient condition for the index to be zero.

R. Finn (Stanford, Calif.)

6478:

Vol'pert, A. I. Boundary value problems for elliptic systems of differential equations of higher order on a plane. Dokl. Akad. Nauk SSSR **127** (1959), 739–741. (Russian)

The results of the preceding paper are here generalized to the problem

$$(1) \qquad \sum_{0 \le k+l \le n} A_{kl} \frac{\partial^{k+l} u}{\partial x^k \partial y^l} = f,$$

$$(2) \qquad \sum_{0 \le k+l \le m_j} a_{kl}^{(j)} \frac{\partial^{k+l} u}{\partial x^k \partial y^l}\Big| = 0 \quad (j = 1, \cdots, \tfrac{1}{2}pn).$$

Under appropriate assumptions on the coefficients, the author proves that the homogeneous problem $(f=0)$ has only a finite number of linearly independent solutions, and that there exists a finite number of linearly independent column vectors v_j, $j=1, \cdots, l$, such that the general problem has a solution if and only if $\iint_D v_j f\,dx\,dy = 0$, $j=1, \cdots, l$. Formulas for the index of the problem (1), (2) and for the difference of this index with the index of the corresponding first boundary problem [see preceding review] are derived. *R. Finn* (Stanford, Calif.)

6479:

Solomyak, T. B. Solution of the first boundary problem for quasi-linear equations of elliptic type containing power-type non-linearities. Dokl. Akad. Nauk SSSR **127** (1959), 274–277. (Russian)

The author studies quasi-linear elliptic equations of the form

$$Lu = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}(a_i(x_j, u, p_j)) + b(x_i, u, p_i) = f(x_i),$$

$$p_i = \frac{\partial u}{\partial x_i}, \quad a_i(x_j, 0, 0) \equiv b(x_j, 0, 0) \equiv 0.$$

Under various assumptions on the coefficients, notably the assumption of a non-linearity such that, for some $n \ge 2$,

$$\sum_{i,j} \frac{\partial}{\partial p_j}(a_i(x_k, u, p_k))t_i t_j \ge \alpha(T^{n-2} + |u|^{n-2} + 1)\sum_i t_i^2 \qquad (T^2 = \sum_i p_i^2),$$

he proves the existence of a unique generalized solution of the first homogeneous boundary value problem in a finite region Ω, and gives an apriori estimate on an integral norm of the solution. Under additional assumptions, estimates are also given for integral norms involving the second derivatives of the solution.

R. Finn (Stanford, Calif.)

6480:

Aronson, D. G. The fundamental solution of a linear parabolic equation containing a small parameter. Illinois J. Math. **3** (1959), 580–619.

Let $x=(x_1, x_2, \cdots, x_n)$ be a point in the real Euclidean n-space E^n and y a positive real variable. Given is the parabolic initial value problem

$$(1) \qquad L_\varepsilon(u) = f(x, y) \quad (y > 0), \quad u(x, 0, \varepsilon) = g(x),$$

where

$$(2) \qquad L_\varepsilon(u) = \varepsilon \sum_{i,j=1}^{n} a_{ij}(x, y) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, y) \frac{\partial u}{\partial x_i} + b(x, y)u - \frac{\partial u}{\partial y},$$

ε is a positive parameter, and the matrix of the a_{ij} is supposed to be positive definite. If $u(x, y, \varepsilon)$ is the solution of (1), the main problem treated in the paper is to prove

$$(3) \qquad \lim_{\varepsilon \to 0^+} u(x, y, \varepsilon) = v(x, y),$$

where $v(x, y)$ is the solution of the reduced initial value problem

$$L_0(v) = f(x, y) \qquad (y > 0), \quad v(x, 0) = g(x)$$

if L_0 denotes the operator obtained from (2) by setting $\varepsilon = 0$.

This problem is attacked as in the following way: in a well-known manner the solution of (1) can be written explicitly in terms of a fundamental solution $\Gamma(x, y; \xi, \eta; \varepsilon)$ of (1). Γ is expressed in terms of a parametrix $G(x, y; \xi, \eta; \varepsilon)$ which satisfies the homogeneous equation $L_\varepsilon = 0$ only approximately but has the same singularities at $x=\xi$, $y=\eta$ as Γ such that certain essential limit relations as $y \to \eta$ for integrals containing Γ remain true if Γ is replaced by G. The author now notices that certain other limit relations for $\varepsilon \to 0$ must be satisfied by Γ if (3) is to be true, but that these relations in general are not true if Γ is replaced by the parametrix used hitherto. However a parametrix is not uniquely determined by the two requirements mentioned above, and the author succeeds in constructing a new parametrix (and through it the fundamental solution) which satisfies also the requirement for $\varepsilon \to 0$ (theorem 1). Using this result he proves (under certain assumptions concerning the given functions) that (3) is true if $v(x, y)$ is the "weak" solution of the reduced problem. Finally estimates concerning the rate of convergence of u to v are obtained (theorem 3).

The convergence $u \to v$ proved in this paper is pointwise. Convergence in the L^2 sense was proved for parabolic systems by O. A. Ladyženskaya [Vestnik Leningrad Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 104–120; MR **19**, 656]. *E. H. Rothe* (Ann Arbor, Mich.)

6481:

Szarski, J. Remarque sur un travail de J. Schauder. Ann. Polon. Math. **6** (1959), 157–160.

The author points out and repairs a gap in J. Schauder's paper, "Anfangeswertproblem einer quasilinearen hyperbolischen Differentialgleichung zweiter Ordung in beliebiger Anzahl von unabhängigen Veränderlichen", Fund. Math. **24** (1935), 213–246.

R. W. McKelvey (Boulder, Colo.)

6482:

Bakel'man, I. Ya. A class of non-linear differential equations. Dokl. Akad. Nauk SSSR **126** (1959), 244–247. (Russian)

Let D be a closed bounded region in n-dimensional Euclidean space with a sufficiently smooth boundary; let $W_2^{(2)}(D)$ be the space of functions with generalized derivatives of order two which belong to $L_2(D)$; for $u(x_1, \cdots, x_n) \in W_2^{(2)}(D)$ let $u_i = \partial u/\partial x_i$, $u_{ik} = \partial^2 u/\partial x_i \partial x_k$, $i \leqq k = 1, \cdots, n$; let Q be a Euclidean space of $l = \frac{1}{2}n(n+1) + n + 1$ dimensjons, $q = (q_{11}, \cdots, q_{nn}, q_1, \cdots, q_n, q)$, where $i \leqq k$ for q_{ik}. For $x \in D$, $u(x) \in W_2^{(2)}(D)$, let $q(x) \in Q$, where $q(x) = (u_{11}(x), \cdots, u_{nn}(x), u_1(x), \cdots, u_n(x), u(x))$. Then the function $F(q, x)$ corresponds to a second order differential operator $F(u)$ and conversely. $F(u)$ is said to be totally elliptic if, for $q \in Q$, $x \in D$, $\sum_{i \leqq k=1}^{n} F_{u_{ik}}(q, x)\xi_i\xi_k \geqq \alpha_0 \sum_{i=1}^{n} \xi_i^2$ or else $-\sum \geqq \alpha_0 \cdot \sum_{i=1}^{n} \xi_i^2$, where the ξ_i's are real and α_0 is a positive constant. Let

$$L(u) = \sum_{i \leqq k=1}^{n} a_{ik}(x)u_{ik}, \quad a_{ik}(x) \in C^1(D),$$

be a linear elliptic operator. Theorem 1: If $F(u)$ is totally elliptic in D and if $L(u)$ is as above, then for each $x \in D$, the surface $F(q, x) = 0$ projects onto the surface $L(q, x) = 0$ in a smooth and unique way. [See also Dokl. Akad. Nauk SSSR **114** (1957), 1143–1145, MR **20** #1983.]

Now let A be the set of linear operators

$$L(u) = \sum_{i \leqq k=1}^{n} a_{ik}(x)u_{ik}$$

for which: (1) $a_{ik}(x) \in C^1(D)$; (2) $\sum_{i \leqq k=1}^{n} a_{ik}^2(x) = 1$; (3) for $x \in D$, ξ_i real,

$$\sum_{i \leqq k=1}^{n} a_{ik}(x)\xi_i\xi_k \geqq \alpha_0 [\sum_{i \leqq k=1}^{n} \hat{a}_{ik}^2(x)]^{-1/2} \cdot \sum_{i=1}^{n} \xi_i^2,$$

where $\hat{a}_{ik}(x) = F_{u_{ik}}(0, x)$, $i \leqq k$. Theorem 2 states that the Dirichlet problem for $F(u) = 0$ has a unique solution in $W_2^{(2)}(D)$ if $F(u)$ is totally elliptic in D and if the norm of F is less than $2^{1/2} \cdot (n^2 + 3n + 2)^{-1/2}$. The norm of F is defined by means of the operators of A. Furthermore an estimate is given for the norm of this solution.

Let $\Phi(u) = H(u) + L(u)$, where $H(u)$ is a polynomial of odd degree in u and its first and second derivatives and $L(u)$ is linear elliptic. Theorem 3 gives conditions on L and H that the Dirichlet problem for $\Phi(u) = 0$ have a unique solution in $W_2^{(2)}(D)$. An estimate for the norm of this solution is also given. No proofs are given. The author states various generalizations of theorems 2 and 3.

C. S. Coleman (Claremont, Calif.)

POTENTIAL THEORY

See also 6425, 6468, 6510.

6483:

Solomencev, E. D. Sur les valeurs limites des fonctions sousharmoniques. Czechoslovak Math. J. **8** (**83**) (1958), 520–536. (Russian. French summary)

Let D be a bounded region in R^n homeomorphic to the unit ball; for $n = 2$ the boundary Γ is assumed to be a Jordan curve with tangent satisfying a Lipschitz condition of some positive order, for $n > 2$ the boundary Γ is assumed to have the property that through each point Q pass two spheres of fixed radius λ, the one lying entirely outside D and the other entirely inside (except for Q). The author first obtains estimates for the Green's function of D near the boundary, then proves the following result: Let u be a subharmonic function in D, μ the associated measure, ψ the derivative of μ relative to Lebesgue measure ω, σ the distance of a point of D from Γ, and B the boundary strip defined by some inequality $0 < \sigma < \sigma_1$. Suppose that: (0) u^+ has a harmonic majorant; (1) for $n = 2$, the integral $\int_B \sigma\psi \ln^+(\sigma\psi) d\omega$ is finite; for $n > 2$, the integral $\int_B (\sigma\psi)(n-1+\alpha)/(1+\alpha) d\omega$ is finite for some α, $0 \leqq \alpha < 1$; (2) almost every point Q of Γ is the vertex of a truncated cone which lies in D, which has the normal to Γ for an axis, which has vertex angle an arbitrary angle less than π, and in which μ is absolutely continuous. Then, at almost every point of Γ, u has a limit along every curve not tangent to Γ, and these limiting values form a summable function. For $n = 2$, these results extend those of Tolsted [Proc. London Math. Soc. **7** (1951), 321–333; MR **20** #1110].

G. A. Hunt (Princeton, N.J.)

6484:

Bouligand, G. Sur quelques problèmes non linéaires. J. Math. Pures Appl. (9) **38** (1959), 201–212.

In the first part of this paper the author discusses the problem of finding a harmonic function $U(P)$ in a region D, for which $|\nabla U|^2$ is prescribed on the boundary. In two dimensions, under suitable restrictions on D and on the prescribed data, the existence of an infinity of solutions can be shown by the simple device of introducing a conjugate harmonic function $V(P)$ and taking logarithms. In higher dimensions, the problem is reduced to an integral equation whose solution is not accessible to standard methods.

The remainder of the paper is devoted to a discussion, largely historical, of the possibility of introducing a general principle into a class of non-linear problems.

R. Finn (Stanford, Calif.)

6485:

Mack, Sidney F. Discontinuities of second derivatives of single-layer potentials. Math. Ann. **137** (1959), 417–426.

The author derives the jump relations (in the Cauchy sense) for the second derivatives of the potential U due to a single-layer distribution on a surface element S. The surface element, with the (x, y)-plane as tangent plane in

the point P under consideration, is assumed to be of class C^2 (whereas former investigations had to require class C^3). The mass distribution must be continuous at P for the (xx)-, (xy)-, (yy)-, and (nn)-derivatives and of class C^1 for the (xn)- and (yn)-derivatives. The proofs are based on the association with S at the point P of an "osculation surface element" S_0. S_0 is tangent to S at P, and its normal sections are circular arcs whose curvature is equal to the normal curvature of S. It turns out that S_0 is of class C^3, but is of class C^4 only if the principal curvatures of S are equal. The jump relations involve the mean curvature of S (for U_{nn}), the normal curvature and the geodesic torsion of the normal section $y=0$ (for U_{xx} and U_{xy}). *J. C. C. Nitsche* (Minneapolis, Minn.)

6486:

Erlich, Amos. Continuation of polyharmonic functions by inversion. Riveon Lematematika **13** (1959), 35–40. (Hebrew. English summary)

Let D be an n-dimensional domain whose boundary contains a portion d lying on the boundary of a sphere S_a with radius a and center $A=(a, 0, \cdots, 0)$. Let $u(q)$ be a p-harmonic function in D and let $u(q)/(a-\rho)^{p-1}\to 0$ as $q\to d$, where ρ is the distance from q to A. It is then proved that $u(q)$ can be continued as a p-harmonic function to the domain $D\cup d\cup D^*$, where D^* is the inverse of D with respect to S_a, and

$$u(q^*)=\left(\frac{a}{\rho}\right)^{2p-n}\sum_{i=0}^{p-1}\frac{(-1)^{p+i}}{(2a)^{2i}(i!)^2}(a^2-\rho^2)^{p+i}r^{-2i}(r^2\Delta r^2)^i \times\left[r^{-2i}\frac{u(q)}{(a^2-\rho^2)^{p-i}}\right].$$

The method is based on that of R. J. Duffin [Duke Math. J. **22** (1955), 313–324; MR **18**, 29] who proved the above result for $n=3$, $p=2$: By inversion the problem is reduced to a similar problem in which S_a is replaced by a hyperplane, and then the result of A. Huber [Pacific J. Math. **5** (1955), 433–439; MR **17**, 28] is applied.

A. Friedman (Minneapolis, Minn.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 6416, 6457, 6702.

6487:

Cooke, K. L. The rate of increase of real continuous solutions of certain algebraic functional equations. Trans. Amer. Math. Soc. **92** (1959), 106–124.

Let $P(t, u, v)$ denote a polynomial in the variables t, u, v with real coefficients and let $\delta(t)$ be positive and continuous for $t\geq a>0$. In this paper the rate of increase of real solutions of algebraic functional equations of the form

$$P(t, u(t), u(t+\delta(t)))=0, \tag{1}$$

when $t\to\infty$, is considered. Equations of this type with $\delta(t)=$ constant were considered earlier by O. E. Lancaster [Bull. Amer. Math. Soc. **46** (1940), 169–177; MR **1**, 181] and S. M. Shah [ibid. **53** (1947), 548–558; Proc. Nat. Inst. Sci. India **16** (1950), 11–17; MR **9**, 189; **12**, 105].

Call b_n the nth δ-image of b if $b_0=b$, $b_{n+1}=b_n+\delta(b_n)$, and let $I_n(b)=[b_n, b_{n+1})$ be called the nth δ-image of the interval $[b, b+\delta(b))$. It is proved that if $t+\delta(t)$ is increasing for $t\geq a$ and $\delta(t)$ also satisfies the condition stated earlier, then there exist positive numbers A and c depending only on the polynomial P such that if $f(t)$ satisfies the two hypotheses noted below, then

$$\liminf_{t\to+\infty}\frac{|u(t)|}{f(t)}=0$$

for every continuous solution $u(t)$ of (1). The two hypotheses are: (2) $f(t)$ is a positive continuous function for $t\geq a$ satisfying $\lim_{t\to+\infty}t^c/f(t)=0$; (3) given any $b\geq a$, any $\omega>1$, and an arbitrary sequence of points $\{\tau_n\}$ having τ_n in the nth δ-image interval $I_n(b)$, then $\lim_{n\to\infty}\omega^{A^n}/f(\tau_n)=0$. Theorems of the form $\lim_{t\to+\infty}|u(t)|/f(t)=0$ for every continuous monotonic solution of (1) have been proved with suitable hypotheses on $f(t)$. Equations of the form $P(t, u(t), u(p+qt))=0$ and $P(t, u(t), u(t^q))=0$ have also been considered. *S. M. Shah* (Evanston, Ill.)

SEQUENCES, SERIES, SUMMABILITY

See also 6411, 6419, 6441, 6457, 6503.

6488:

Alexiewicz, A.; and Orlicz, W. On summability of double sequences. II. Ann. Polon. Math. **6** (1959), 171–180.

This article represents a continuation of an earlier paper by the same authors [same Ann. **2** (1955), 170–181; MR **18**, 205]. We recall some standard notions. By a convergent sequence $x=\{x_{ik}\}$ we shall mean convergence in Pringsheim's sense. Denote by $\mathrm{Lim}_{i,k\to\infty}x_{ik}$ the limit of the sequence. Denote by S_n the set of pairs (i, k) of indices such that $n^{-1}\leq(i+1)(k+1)^{-1}\leq n$. Then the sequence is called restrictedly convergent to $x_{..}$ [Moore, Math. Ann. **74** (1913), 555–572] if given $\varepsilon>0$ and n there is a N such that $i, k>N$ together with $(i, k)\in S_n$ implies $|x_{ik}-x_{..}|<\varepsilon$. This restricted limit is denoted by $[\mathrm{Lim}_{i,k\to\infty}]x_{ik}$. Let $A=(a_{ikuv})$ be a four-dimensional matrix. Consider the transforms $A_{ik}(x)=\sum_{u,v=0}^{\infty}a_{ikuv}x_{uv}$. If the series converge in Pringsheim's sense and $\mathrm{Lim}_{i,k\to\infty}A_{ik}(x)$ exists and equals $A_{..}(x)$, then x is called A-summable to $A_{..}(x)$. If only $[\mathrm{Lim}_{i,k\to\infty}]A_{ik}(x)$ exists the sequence is called restrictedly A-summable. By BC and $[BC]$ we denote the class of all bounded convergent and bounded restrictedly convergent sequences. Let z be any class of bounded convergent or bounded restrictedly convergent sequences. The method A is called restrictedly conservative or r-conservative for z if it transforms every sequence of z into a bounded restrictedly convergent sequence. The method is called r-permanent if it is r-conservative and the generalized limit $[A]_{..}(x)$ is equal to the ordinary limit. An application of two norm spaces yields the following principal theorem. Let the methods A and B be r-permanent for BC_0 and let every bounded sequence $x=\{x_{ik}\}$ restrictedly A-summable to zero be restrictedly B-summable. Then B-$[\lim_{i,k\to\infty}x_{ik}]=0$. In this theorem C_0 consists of the null-convergent sequences.

V. F. Cowling (Lexington, Ky.)

6489:

Parameswaran, M. R. On summability functions for the circle family of methods. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 171–175.

This note is based on the idea of summability functions, introduced by G. G. Lorentz [Canad. J. Math. **1** (1949), 305–319; MR **11**, 242]. The author is concerned with four types of regular matrices: the Euler-Knopp matrix

$$E(p) = \binom{n}{k} p^k (1-p)^{n-k},$$

the Taylor matrix

$$T(p) = \binom{k}{n} p^{n+1} (1-p)^{k-n},$$

the Laurent matrix

$$L(p) = \binom{n+k}{k} p^{n+1} (1-p)^k \quad (n, k = 0, 1, 2, \cdots),$$

and the F_α matrix [see, e.g., Hardy, *Divergent series*, Clarendon Press, Oxford, 1949; MR **11**, 25]. The following results are proved. Theorem 1: All functions $\Omega(n) = o(n^{1/2})$ are summability functions of the first kind and of the second kind for the "circle family" $\mathfrak{C}$ of methods, i.e., for the methods $E(p)$, $T(p)$, $L(p)$, and F_α, with $0 < p < 1$, $\alpha > 0$. The result for $E(p)$ had already been established by Lorentz in the paper cited above. Theorem 2: Let A belong to the class $\mathfrak{C}$ (the circle family); then every summability function for the method A is of the form $\Omega(n) = o(n^{1/2})$. The next result is concerned with members of $\mathfrak{C}$ which are absolutely translative (or absolutely regular) [see the reviewer's *Infinite matrices and sequence spaces*, Macmillan, London, 1950; MR **12**, 694; cf. p. 114 and p. 119]. Theorem 3: Let A be a member of the class $\mathfrak{C}$; then A is absolutely translative for all sequences $\{\sigma_n\}$ for which $\sigma_n = o(n^{1/2})$. Theorem 4: If $\{\sigma_n\}$ is a sequence for which

$$\frac{\sigma_0 + \sigma_1 + \cdots + \sigma_n}{n+1} = l + o(n^{-1/2}) \quad (l \text{ finite}),$$

then $\{\sigma_n\}$ is summable to l by every member of the class $\mathfrak{C}$. Further, the function $o(n^{-1/2})$ cannot be replaced by any function $\phi(n) \neq o(n^{-1/2})$ for any member of the class $\mathfrak{C}$.

R. G. Cooke (London)

6490:

Agnew, R. P. Riemann summability of Cauchy products and translates of series. J. Indian Math. Soc. (N.S.) **22** (1958), 33–44.

Der Verf. konstruiert für jedes $p = 1, 2, 3, \cdots$ eine $R(p)$ (Riemann-Summierbarkeit) summierbare Reihe $\sum_0^\infty u_n$, so dass weder $0 + u_0 + u_1 + \cdots$ noch $u_1 + u_2 + u_3 + \cdots$ $R(p)$-summierbar ist. Für den Fall $p = 2$ wurde ein derartiges Beispiel ohne Beweis von Kuttner [J. London Math. Soc. **11** (1936), 301–302] angegeben. Durch diese Beispiele wird eine Behauptung von Vignaux [C. R. Acad. Sci. Paris **195** (1932), 750–751] widerlegt, nach der $\sum_0^\infty |v_n| < \infty$ und die $R(p)$-Summierbarkeit von $\sum_0^\infty u_n$ die $R(p)$-Summierbarkeit von $\sum_{n=0}^\infty \sum_{p+q=n} u_p v_q$ nach sich zieht; daraus würde nämlich mit $v_1 = 1$, $v_i = 0$ für $i \neq 1$ die $R(p)$-Summierbarkeit von $0 + u_0 + u_1 + \cdots$ folgen.

A. Peyerimhoff (Marburg)

6491:

Flett, T. M. Some remarks on strong summability. Quart. J. Math. Oxford Ser. (2) **10** (1959), 115–139.

The general theory of strong summability is discussed. In particular, the author gives a new definition of strong Abel summability and considers the relation of strong Cesàro summability to it. It is said that $\sum_{n=0}^\infty a_n$ is strongly summable (A, k) to the sum s, if the sum $\sum_{n=0}^\infty a_n x^n$ converges for $|x| < 1$ and its sum function $\varphi(x)$ satisfies the condition

$$\int_0^R |\varphi(x) - s|^k (1-x)^{-2} dx = o\{(1-R)^{-1}\}$$

as $R \to 1-0$. The relation of strong Cesáro summability to strong Abel summability is similar to that of ordinary Cesàro summability to ordinary Abel summability. The relation of strong summability to absolute summability is also discussed. In the last section, the author gives a simple theorem on the function $s(\theta)$ of Lusin [A. Zygmund, *Trigonometric series*; #6498 below; p. 207], but this is independent of strong summability.

G. Sunouchi (Evanston, Ill.)

APPROXIMATIONS AND EXPANSIONS

See also 6442, 6443a–b, 6444, 6681, 6682, 6683.

6492:

Egerváry, E.; and Turán, P. Notes on interpolation. VI. On the stability of the interpolation on an infinite interval. Acta Math. Acad. Sci. Hungar. **10** (1959), 55–62. (Russian summary, unbound insert)

As in part V [same Acta **9** (1958), 259–267; MR **21** #2136] the authors consider interpolating polynomials of the form $\sum_{\nu=1}^n y_\nu r_{\nu n}(x)$, y_ν given real values, $r_{\nu n}(x)$ given polynomials, satisfying the conditions $r_{\nu n}(x_{jn}) = \delta_{\nu j}$ where $x_{\nu n}$ are prescribed, increasing with ν. In the previous paper the case of a finite interval was considered, in the present one the intervals (a) $0, +\infty$ and (b) $-\infty, +\infty$; the $x_{\nu n}$ are situated in the respective intervals. The interpolation is called stable if for any systems y_ν, y_ν^* where $y_\nu \geqq y_\nu^*$, the following is true:

$$\text{(a)} \qquad 0 \leqq e^{-x}\left(\sum_{\nu=1}^n y_\nu r_{\nu n}(x) - \sum_{\nu=1}^n y_\nu^* r_{\nu n}(x) \right) \leqq \max (y_\nu - y_\nu^*) e^{-x_{\nu n}},$$

$$\text{(b)} \qquad 0 \leqq e^{-x^2}\left(\sum_{\nu=1}^n y_\nu r_{\nu n}(x) - \sum_{\nu=1}^n y_\nu^* r_{\nu n}(x) \right) \leqq \max (y_\nu - y_\nu^*) e^{-x_{\nu n}^2},$$

respectively. The main results are the following: For a stable procedure the quantity $\sum_{\nu=1}^n$ (degree of $r_{\nu n}(x)$) is a minimum if and only if: in the case (a), $x_{1n} = 0$ and the further $x_{\nu n}$ are the zeros of the Laguerre polynomial $L_{n-1}(x)$; in the second case, (b), the $x_{\nu n}$ must be the zeros of the Hermite polynomial $H_n(x)$.

G. Szegö (Stanford, Calif.)

6493:

Balázs, J.; and Turán, P. Notes on interpolation. VII. Convergence in infinite intervals. Acta Math. Acad. Sci. Hungar. **10** (1959), 63–68. (Russian summary, unbound insert)

In part VI [review above] the following polynomials of interpolation were introduced:

$$R_n(f) = \sum_{\nu=1}^n f(x_\nu) \frac{x}{x_\nu} \left(\frac{L_{n-1}(x)}{L'_{n-1}(x_\nu)(x - x_\nu)} \right)^2,$$

where L_{n-1} is Laguerre's polynomial, with zeros $x_2, \cdots, x_n$, and for $\nu=1$ the fundamental polynomial has to be modified to $[L_{n-1}(x)]^2$. The authors prove that if $f(x)$ is continuous and bounded in $[0, \infty)$ then $R_n(f)\to f(x)$ in $[0, \infty)$, uniformly in every finite interval. A similar result holds for the polynomials of interpolation introduced by Egerváry and Turán on the zeros of Hermite polynomials.

G. Szegö (Stanford, Calif.)

6494:

Askey, Richard; and Hirschman, Isidore, Jr. Weighted quadratic norms and ultraspherical polynomials. I. Trans. Amer. Math. Soc. **91** (1959), 294–313.

If $\nu\geq 0$, the ultraspherical polynomials $W(n, x)=W_\nu(n, x)$ of degree n and index ν are defined by

$$2^n\Gamma(\nu+n+1/2)W_\nu(n, x) = n!(-1)^n(1-x^2)^{-\nu+1/2} \times \left(\frac{d}{dx}\right)^n [(1-x^2)^{n+\nu-1/2}]$$

and have the orthogonality property

$$\omega(n)\int_{-1}^{1} W(n, x)W(m, x)d\Omega(x) = 1 \text{ or } 0$$

according as $n=m$ or $n\neq m$, where $d\Omega(x)=d\Omega_\nu(x)=(1-x^2)^{\nu-1/2}$ and $\omega(n)=\omega_\nu(n)$ is independent of x. A function $f(x)$ on $[-1, 1]$ belongs to $B=B_\nu$ if $\int_{-1}^{1}|f(x)|d\Omega(x)<\infty$, and has a formal expansion in terms of $W(n, x)$ given by

$$f(x) = \sum_{n=0}^{\infty} f\hat{}(n)W(n, x)\omega(n),$$

where the coefficients $f\hat{}(n)$ are defined by

$$f\hat{}(n) = \int_{-1}^{1} f(x)W(n, x)d\Omega(x) \quad (n = 0, 1, 2, \cdots).$$

On the other hand, if $\sum_0^\infty |F(n)|\omega(n)<\infty$, we can define

$$F\hat{}(x) = \sum_{0}^{\infty} F(n)W(n, x)\omega(n),$$

and derive the dual formula

$$F(n) = \int_{-1}^{1} F\hat{}(x)W(n, x)d\Omega(x).$$

If $f_1(x)$, $f_2(x)$ belong to B, their convolution $f_1*f_2(x)$ is defined in terms of a certain kernel function $C(x, y, z)=C_\nu(x, y, z)$ by

$$f_1*f_2(x) = \int_{-1}^{1}\int_{-1}^{1} f_1(y)f_2(z)C(x, y, z)d\Omega(y)d\Omega(z),$$

and it is shown that f_1*f_2 then belongs to B and has coefficients

$$f_1*f_2\hat{}(n) = f_1\hat{}(n)f_2\hat{}(n).$$

In the dual case, the convolution $F_1*F_2(n)$ of the sequences $F_1(n)$, $F_2(n)$ is defined in terms of a form $c(k, j, n)$ by

$$(1) \quad F_1*F_2(n) = \sum\sum F_1(k)F_2(j)c(k, j, n)w(k)w(j),$$

and it then follows that $F_1*F_2\hat{}(x)=F_1\hat{}(x)F_2\hat{}(x)$. The functions $f(x)$ on $[-1, 1]$ which satisfy

$$\mathfrak{N}_{\beta,\alpha}[f] = \left[\int_{-1}^{1} f(x)^2(1+x)^\beta(1-x)^\alpha d\Omega(x)\right]^{1/2} < \infty$$

form a linear metric space $\mathfrak{N}_{\beta,\alpha}$ with $\mathfrak{N}_{\beta,\alpha}[f]$ as metric, and the main object of the paper is to determine sufficient conditions to ensure the boundedness of a linear transformation T of $\mathfrak{N}_{\beta,\alpha}$ into itself when T is defined by $(Tf)\hat{}(n)=f\hat{}(n)t(n)$ and so commutes with the convolution operator in the sense that $(Tf_1)*f_2=f_1*(Tf_2)$. The main conclusion is that T has a finite bound $CA(\alpha, \beta, \nu)$ provided that $-\frac{1}{2}<\alpha, \beta<\frac{1}{2}$ and $|t(n)|\leq C$, $\sum|t(k)-t(k-1)|\leq C$ for sums over $2^n\leq k<2^{n+1}$. This result follows interesting relationships between weighted quadratic norms of $f(x)$ and series involving its coefficients $f\hat{}(n)$. Thus, if

$$s(x) = \sum_{k=1}^{\infty}[1-W(k, x)]r(k), \quad S(m, n) = \sum c(m, n, k)r(k),$$

$r(k)\geq 0$, $\sum r(k)<\infty$, it is shown that

$$(2) \quad \int_{-1}^{1} f(x)^2 s(x)d\Omega(x) = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[f\hat{}(n)-f\hat{}(m)]^2S(m, n)\omega(n)\omega(m)$$

when the left-hand side is finite. In the special case $r(k)=k^{-2\alpha-1}$ $(\alpha>0)$, the left-hand side lies between positive constant multiples of $\mathfrak{N}_{0,\alpha}[f]^2$. Further, if $0\leq\alpha\leq\frac{1}{2}$ then

$$\sum f\hat{}(n)^2\omega(n)[R(n)+1]^{-2\alpha} \leq A(\alpha)\mathfrak{N}_{0,\alpha}[f]^2,$$

$$\mathfrak{N}_{0,\alpha}[S_N] \leq A(\alpha)\mathfrak{N}_{0,\alpha}[f]$$

if $R(0)$, $R(1)$, $\cdots$ is any rearrangement of 1, 2, $\cdots$ and

$$S_N = \sum_{0}^{N} f\hat{}(n)\omega(n)W(n, x).$$

The dual convolution structure is due to Lewitan [Mat. Sb. (N.S.) **17** (**59**) (1945), 9–44; MR **7**, 254] and Bochner [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1141–1147; MR **16**, 834] and the paper and its sequel complete the programme initiated by Hirschman [Canad. J. Math. **7** (1955), 462–482; MR **17**, 477]. *H. R. Pitt* (Nottingham)

6495:

Hirschman, I. I., Jr. Weighted quadratic norms and ultraspherical polynomials. II. Trans. Amer. Math. Soc. **91** (1959), 314–329.

The results here can be regarded as dual to those in the preceding paper. A transformation t of sequences $F(n)$ which commutes with the convolution operator defined in (1) above in the sense that $tF_1*F_2=F_1*tF_2$ determines a measurable function $t(x)$ on $[-1, 1]$ so that

$$[tF]\hat{}(x) = F\hat{}(x)t(x).$$

The sequences $F(n)$ which satisfy

$$\mathfrak{N}_\alpha[F] = [\sum F(n)^2\omega(n)(n+1)^{2\alpha}]^{1/2}<\infty \quad (-1/2<\alpha<1/2)$$

form a linear metric space $\mathfrak{N}_\alpha$ with $\mathfrak{N}_\alpha[f]$ as metric. If $p(x)\geq 0$, the duals $q(n)$, $Q(x, y)$ of $s(x)$ and $S(m, n)$, respectively, are defined by

$$q(n) = \int_{-1}^{1}[1-W(n, x)]p(x)dx,$$

$$Q(x, y) = \int_{-1}^{1} C(x, y, z)p(z)dz$$

and the dual of (2) then becomes

$$\sum_0^\infty F(n)^2 q(n)\omega(n) = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}[F\hat{}(x)-F\hat{}(y)]^2 Q(x,y)d\Omega(x)d\Omega(y).$$

This result is used to show that

$$\int_0^\pi\int_0^\pi [F\hat{}(\cos\theta)-F\hat{}(\cos\phi)]^2|\theta-\phi|^{-2\alpha-1} \times [1-\cos(\theta+\phi)]^{-\nu}\sin^{2\nu}\theta\sin^{2\nu}\phi\,d\theta d\phi$$

lies between positive constant multiples of $\mathfrak{R}_\alpha[F]^2-\omega(0)^2F(0)^2$ if $0<\alpha<1/2$. Finally, a transformation t of $\mathfrak{R}_\alpha$ into itself has a bound $CA(\alpha,\nu)$ provided that $|\tau(\theta)|=|t(\cos\theta)|\leq C$ for $0\leq\theta\leq\pi$ and $\int|d\tau(\theta)|\leq C$ when the integral is taken over intervals with end points $\frac{1}{2}\pi[1\pm 2^{-k}]$, $\frac{1}{2}\pi[1\pm 2^{-k+1}]$ ($k=1, 2, 3, \cdots$). This is an analogue of the main result of part I. *H. R. Pitt* (Nottingham)

6496:

Verblunsky, S. On an expansion in exponential series. II. Quart. J. Math. Oxford Ser. (2) **10** (1959), 99–109.

The author continues his previous investigation [same J. **7** (1956), 231–240; MR **20** #4129]. For the kind of exponential expansion considered (connected with a given function $k(u)$, of bounded variation in $(0, 1)$; see the above-quoted review) he proves an analogue of the Riesz-Fischer theorem and (under certain restrictions which cannot be avoided) its converse. Further he considers the integral equation $\int_0^1 k(u)f(x+u)du=g(x)$, where g is given for all x, f is given in $(0, 1)$, and it is required to define f outside this interval so as to satisfy the equation. It is assumed that $f(x)$ is continuous and of bounded variation in $(0, 1)$, that $g'(x)$ has the same property in every finite interval, and that the (necessary) initial conditions $\int_0^1 k(u)f(u)du=g(0)$, $\int_0^1 k(u)df(u)=g'(0)$ are satisfied. The demand that the solution $f(x)$ be continuous and of bounded variation in every finite interval secures uniqueness of the solution. An explicit expression for $f(x)$ is given. *E. Følner* (Copenhagen)

6497:

Korovkin, P. P. Best approximation of functions of class Z_2 by certain linear operators. Dokl. Akad. Nauk SSSR **127** (1959), 513–515. (Russian)

The operators in question are

$$L_r(f,x) = \pi^{-1}\int_{-\pi}^{\pi} f(x+t)u_r(t)dt,$$

where $u_r(t)=\frac{1}{2}+r\cos t+\sum_{k=2}^\infty \varphi_k(r)\cos kt$ $(0<r<1)$ satisfies $u_r(t)\geq 0$, and $\sum|\varphi_k(r)|<+\infty$. A 2π-periodic function f belongs to the class Z_2 if $|f(x+t)+f(x-t)-2f(x)|\leq t^2$. Let $a_r=\sup_{f\in Z_2}\max_x|L_r(f,x)-f(x)|$, and let b_r denote the infimum of a_r for all admissible functions $u_r(t)$. Then $\lim_{r\to 1}(1-r)^{-1}b_r=1$. The optimal function $u_r(t)$ is given by $\frac{1}{2}+\sum_{k=1}^\infty r^{k^2}\cos kt$. For this $u_r(t)$, the asymptotic value of a_r is also determined when Z_2 is replaced by the class of functions f with

$$|f(x+t)+f(x-t)-2f(x)| \leq 2|t|.$$

G. G. Lorentz (Syracuse, N.Y.)

FOURIER ANALYSIS

See also 6516.

6498:

★Zygmund, A. Trigonometric series. 2nd ed. Vols. I, II. Cambridge University Press, New York, 1959. Vol. I. xii+383 pp.; Vol. II. vii+354 pp. $15.00 per vol.; $27.50 the set.

The volumes under review comprise a complete revision, with many amplifications and improvements, of the classic first edition [*Trigonometrical series*, Monografje Matematyczne, Tom V, Warszawa-Lwów, 1935]. It is a book of almost overwhelming scholarship, and yet it is written in so lucid a style that reading it is a positive pleasure. The author indulges in occasional asides and footnotes which point out the ideas behind proofs and illuminate obscure technical points. He frequently gives two proofs of important theorems—a real variable and a complex variable proof, say, or a conformal mapping and a non-conformal mapping proof. All chapters but one end with a section called "Miscellaneous theorems and examples", which range from simple exercises to quite difficult theorems (frequently hints are given). For every chapter, there are notes assembled at the end of the volume giving historical background, references for further reading, and often statements of results not included in the main text. There is a brief and probably inadequate index.

As in the first edition, the author's viewpoint is that of a classical analyst. Every conceivable theorem, technique, method, device, and trick from the theory of functions both of a real and complex variable is used with a master's hand. Banach space methods are also employed, as in the first edition, with telling effect. But the currently popular techniques of group theory and distributions receive only a passing mention, in the preface. Banach algebras are mentioned nowhere. The viewpoint implicit in this selection of techniques is a legitimate one, amply justified by the fantastically precise and beautiful results obtained by classical methods.

We proceed to outline the topics covered. Chapter I: Trigonometric series and Fourier series. Auxiliary results. Here are given basic definitions, facts about orthogonal functions, a description of L^r spaces, inequalities, and various needed topics from real function theory. (Other background material—modulus of continuity, Toeplitz summability of sequences, fractional integrals, parts of complex function theory—appears later in the book, being presented as needed.) Chapter II: Fourier coefficients. Elementary convergence theorems. Chapter III: Summability of Fourier series. Chapters II and III differ little from the corresponding chapters of the first edition, except for added explanations and some new material (e.g., uniform approximation of continuous functions by trigonometric polynomials).

Chapter IV: Classes of functions and Fourier series. Here important changes from the first edition appear. A beautiful real variable proof due to Marcinkiewicz is given for the existence of the conjugate function; the Riesz-Fischer theorem is given, as in the first edition; Banach space methods are explained and used; Orlicz spaces are studied; and the usual theorems on multipliers for various classes of Fourier series are given. The Tauberian theorem for the circle group appears as an exercise

(p. 180, no. 12). Chapter V: Special trigonometric series. Here the author discusses: series with monotone coefficients; various specific series; Fourier-Stieltjes transforms of measures with carriers contained in certain perfect nowhere dense sets; Riesz products; some theorems on lacunary series; applications of Rademacher functions. Chapter VI: The absolute convergence of Fourier series. The principal addition to the corresponding chapter of the first edition is a detailed discussion of sets on which Fourier series can converge absolutely without converging absolutely everywhere.

Chapter VII: Complex methods in Fourier series. This chapter has been greatly amplified and revised. A short and easy proof (a revision of that found in the first edition) is given for the existence of the conjugate function. An elegant proof due to Calderón is given for M. Riesz's theorem on conjugate functions in L^p $(p>1)$. It seems not to be pointed out explicitly that the characteristic function of the non-negative integers is a multiplier for the class L^p, a fact that is a highly interesting consequence of M. Riesz's theorem. A detailed analysis of the complex function classes H^p and N is given, which is easily the best description of the subject that the reviewer has seen. This culminates in the famous theorem of F. and M. Riesz that a Fourier-Stieltjes transform vanishing for negative integers is the transform of an absolutely continuous measure. (The theorem is not stated in quite this form.)

Chapter VIII: Divergence of Fourier series. As in the first edition, this chapter deals with Lebesgue's, Fejér's, and Kolmogorov's examples of divergent Fourier series, all simplified as much as seems possible. (Gibbs' phenomenon is now treated in chapters II and III.) Chapter IX: Riemann's theory of trigonometric series. This is chapter XI of the first edition, with many additions. The new version is vastly easier to follow. Chapter X: Trigonometric interpolation. This chapter is new. It deals in great detail with the classical problem of approximating functions by their interpolating trigonometric polynomials. The viewpoint is elementary in the sense that only Riemann integration is used (ordinarily finite sums written as Riemann-Stieltjes integrals). The analogy with Fourier expansions is noted and utilized. Chapter XI: Differentiation of series. Generalized derivatives. This chapter, also new, deals with summability by various methods of differentiated Fourier series, along with generalized derivatives of real functions, Denjoy integrals, and the application of these notions to Fourier series.

Chapter XII: Interpolation of linear operations. More about Fourier coefficients. This chapter is based on chapter IX of the first edition but is so changed as to be barely recognizable. The Riesz-Thorin convexity theorem is first proved; the proof, based on the maximum modulus principle, is due to the author and Calderón. It is applied, as usual, to the Young-Hausdorff inequality. Interpolation of linear operators in H^p spaces is discussed. A theorem of Marcinkiewicz on interpolation of operators that are not quite linear and are continuous only in a restricted sense is proved. This powerful result is applied to prove Paley's theorem on Fourier coefficients of functions in L^p. This new proof of Paley's theorem is much simpler than the original one. Fractional integrals, the class of measures whose Fourier-Stieltjes coefficients are $o(1)$, and other topics about Fourier-Stieltjes coefficients are also discussed.

Chapter XIII: Convergence and summability almost everywhere. Some of the results given here appear in various places in the first edition, while others are new. The following matters are treated: conditions on coefficients and on the function itself sufficient for convergence almost everywhere of a Fourier series; Hardy and Littlewood's theory of strong summability; almost convergence of Fourier series; Paley's theorem on the suprema of partial sums of expansions in orthogonal functions (again Paley's original proof is made much more readable; however the recent proof by Stein and Weiss [J. Math. Mech. **8** (1959), 263–284; MR **21** #5888] is not given, although the paper is listed in the bibliography); capacity of sets and convergence of Fourier series.

Chapter XIV: More about complex methods. This chapter contains a number of very delicate results about the boundary behavior of harmonic and analytic functions in the open unit disk, with applications to functions in H^r and to conjugate series.

Chapter XV: Applications of the Littlewood-Paley function to Fourier series. A main result of this chapter is the fact (due to Hardy and Littlewood) that lacunary partial sums of the Fourier series of any function in L^r $(r>1)$ converge almost everywhere. The analysis involved in establishing this and related facts is extraordinarily refined.

Chapter XVI: Fourier integrals. This chapter gives a brief but remarkably complete treatment of Fourier integrals $(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ixy}f(x)dx$ for f in various function classes, and also an introduction to Fourier-Stieltjes transforms. The analogy with Fourier series is used to carry series proofs over to the case of integrals, wherever possible. Also parts of the analytical machinery developed previously, such as the Riesz-Thorin theorem and the theory of the conjugate function, can be re-used here without modification. Topics treated are: the usual convergence and inversion theorems; Plancherel's theorem; the Young-Hausdorff inequality for transforms of functions in L^p $(1<p<2)$; Hilbert transforms (for these last two topics, it is instructive to compare the author's treatment with Titchmarsh's [*Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford, 1937; chapters III–V]); the author's theorem that the Fourier integral of a function in L^p $(1<p<2)$ converges almost everywhere; Fourier-Stieltjes transforms and distribution functions; the Paley-Wiener theorem; Riemann theory of trigonometric integrals.

Chapter XVIII: A topic in multiple Fourier series. This chapter is an introduction to a large and as yet largely unexplored theory. The author carefully points out similarities and differences between the one- and many-variable cases; theorems that go over from one to many variables are of course proved. Mostly rectangular partial sums are employed, since these seem best fitted to $(C, 1)$ and Abel means. Restricted summability of arbitrary Fourier series is proved. The chapter, and book, close with an exposition of results of the author and Calderón on the boundary behavior of power series in several complex variables.

Not every topic in harmonic analysis for the circle, the integers, and the line can be treated even in a book of this size. The author has exercised a perfectly justified liberty in choosing his topics as he has, and many of them he has put into final form. There seems nothing more to say about theorems of the Riesz-Thorin type, for example, and no one surely will need to know more about convergence and

summability of lacunary partial sums of Fourier series and conjugate series than is found in chapter XV. Exercising a similar liberty, the reviewer may be permitted to regret the absence of such topics as Wiener's Tauberian theorem (which is mentioned only in an exercise), spectral synthesis, and multiplicative linear functionals on the algebra of Fourier-Stieltjes coefficients. All of these topics are well enough established and sufficiently classical to warrant inclusion in a book like the present one.

A more fundamental complaint is that the author loses some unity in his subject by not thinking of groups and character groups. The circle, the integers, the line, the direct product of $\aleph_0$ 2-element groups (of which the Walsh-Rademacher functions are the characters), and the n-dimensional torus are the groups he works with. For the reviewer, at least, Fourier series, Fourier-Stieltjes series, Fourier integrals, Fourier-Stieltjes transforms, and expansions in Rademacher functions gain greatly in perspicuity when thought of as mappings of a function or measure space on a group to a function space on the character group. The author hints at this viewpoint in writing on p. 102, vol. II, that "It is apparent that between the two parts of the Hausdorff-Young theorem there is a certain dualism ... This dualism can be detected in various parts of the theory of Fourier series and is an important guide in the search for new results." From the abstract point of view, the two parts of the Hausdorff-Young theorem are special cases of one general theorem, dealing with an arbitrary locally compact Abelian group and its character group. The guide to new results is the tactical axiom that if a theorem is true for the circle and for the integers, then it should be investigated for all locally compact Abelian groups. It also seems clear that the notions and techniques of abstract measure theory are of value in studying Fourier-Stieltjes transforms. To cite one example, consider the construction given by Yu. A. Šreĭder of singular measures on the circle and line [Mat. Sb. (N.S.) **27** (**69**) (1950), 297–318; MR **12**, 420].

Every analyst should be familiar with, and every harmonic analyst should carefully study, this book. In spite of the vast scope of the work, Professor Zygmund's complete mastery of his subject, and his charming style (reminiscent of Hardy's), have made it eminently readable. The beautiful presswork will add to the reader's enjoyment. It can be taken, if the extensive cross-references are used, in almost any order. The preface is an essay that will repay careful study.

It seems safe to predict, indeed, that as analysts for 25 years have been reared on the first edition of Zygmund's book, so analysts for another quarter century will be brought up on the second. *E. Hewitt* (Seattle, Wash.)

6499:

Helson, Henry. Conjugate series in several variables. Pacific J. Math. **9** (1959), 513–523.

Pursuing his work in a recent paper [same J. **8** (1958), 437–446; MR **20** #5397], the author extends some of the well-known inequalities for conjugate functions [A. Zygmund, *Trigonometrical series*, Subwencji Funduszu Kultury Narodwej, Warsaw-Lwów, 1935; Ch. 7] from one variable to several and, more generally, to an infinite number of variables, by adopting a method first introduced by S. Bochner [Ann. of Math (2) **40** (1939), 769–799].

K. Chandrasekharan (Bombay)

6500:

Še, Din-fan. Absolute convergence of Fourier series. Acta Math. Sinica **9** (1959), 199–212. (Chinese. Russian summary)

Let $f(x)$ be a function integrable L^p $(1<p\leqq 2)$ over $(0, 2\pi)$. Let

$$f(x) \sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

Set

$$\Delta_h^{(l)}f(x)=\sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}f(x+jh),$$

$$A_n^{(l)}(N,f)_p=\left\{N^n\int_0^{2\pi}|\Delta_{\pi/N^n}^{(l)}f(x)|^p dx\right\}^{1/p},$$

where $N>1$ and n and l are two natural numbers. The author proves that if the series

$$\sum_{n=1}^{\infty}\{A_n^{(l)}(N,f)_p\}^\beta N^{n(\gamma+1-\beta)}$$

converges, where $\gamma\geqq 0$, $0<\beta<p/(p-1)$, then the series

$$\sum_{n=1}^{\infty}(|a_n|^\beta+|b_n|^\beta)n^\gamma$$

converges. This result contains a known theorem of S. B. Stečkin [cf. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 221–246; MR **17**, 149]. The above theorem can also be extended to the double Fourier series by considering the convergence of the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(|a_{m,n}|^\beta+|b_{m,n}|^\beta+|c_{m,n}|^\beta+|d_{m,n}|^\beta)(m+1)^\gamma(n+1)^{\gamma'},$$

where $a_{m,n}, b_{m,n}, c_{m,n}$ and $d_{m,n}$ are the Fourier coefficients of $f(x, y)$ (a function of L^p) and $\gamma\geqq 0$, $\gamma'\geqq 0$, $0<\beta<p/(p-1)$.

Fu Cheng Hsiang (Taipeh)

6501:

Yano, Kenji. Notes on uniform convergence of trigonometrical series. I. Proc. Japan Acad. **35** (1959), 197–200.

In a previous paper [Tôhoku Math. J. (2) **10** (1958), 19–31; MR **20** #1870] the author studied the uniform convergence of the series

$$\sum_1^{\infty}n^{-1}s_n\sin nt,\quad \sum_1^n a_\nu=s_n,$$

concerning the Riemann summability R_1. In this note similar results are proved for the cosine-analogue. Let $s_n^\gamma=\sum_0^n A_{n-\nu}^{\gamma-1}s_\nu$ with $A_n^\gamma=\binom{\gamma+n}{n}$, $r>0$, $0<s<1$ (or $s=1$, $2, \cdots$), $0<\alpha\leqq 1$, and (1) $\sum_1^n|s_\nu r|=o(n^{1+r\alpha})$, (2) $\sum_n^{2n}(|s_\nu{}^{-s}|-s_\nu{}^{-s})=O(n^{1-s\alpha})$, $n\to\infty$, then (I) for $0<\alpha<1$ the series (*) $\sum n^{-1}s_n\cos nt$ converges uniformly and (II) for $\alpha=1$ the series (*) converges uniformly if and only if $\sum_1^\infty n^{-1}s_n$ converges. The case $r=0$ is treated separately, and (1) and (2) in (II) should be replaced by $\sum_n^{2n}|s_\nu|=o(n/\log n)$, $\sum_n^{2n}(|s_\nu{}^{-s}|-s_\nu{}^{-s})=O(n^{1-\delta})$, $n\to\infty$ $(s>0,\ \delta>0)$.

M. Tomić (Belgrade)

6502:

Sunouchi, Gen-ichirô. On the Riesz summability of Fourier series. Tôhoku Math. J. (2) **11** (1959), 321–326.

Let $f(x)$ be integrable in Lebesgue's sense and periodic with period 2π. Let

(a) $$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Wang [Proc. London Math. Soc. (2) **51** (1949), 215–231; MR **11**, 27] has proved that if $1<\alpha<2$ and the series

$$\sum_{n=2}^{\infty} (a_n^2+b_n^2)(\log n)^{\alpha-1}$$

converges, then the series (a) is summable $(R, \exp(\log n)^{\alpha}, \delta)$ almost everywhere for any positive δ. The author improves Wang's result as follows. Theorem A: If $1<\alpha<\infty$ and the series

$$\sum_{n=2}^{\infty} (a_n^2+b_n^2)|\log(\log n)|$$

converges, then the series (a) is summable $(R, \exp(\log n)^{\alpha}, \delta)$ almost everywhere for any positive δ. Theorem B: If $0<\alpha\leqq 1$ and the series

$$\sum_{n=2}^{\infty} (a_n^2+b_n^2)(\log n)^{\alpha}$$

converges, then the series (a) is summable $(R, \exp|\exp(\log n)^{\alpha}|, \delta)$ almost everywhere for any positive δ. The case $\alpha=1$ of theorem B is the theorem of Kolmogoroff-Seliverstoff-Plessner. *Fu Cheng Hsiang* (Taipeh)

6503:

Lee, Ching-Hsi. On the Gibbs phenomenon for Riemann summation (R, k) of generalized Fourier series. Acta Math. Sinica **9** (1959), 28–36. (Chinese. English summary)

Let k be a positive integer. $\sum_{n=1}^{\infty} u_n$ is said to be summable (R, k) to the sum S if $\sum_{n=1}^{\infty} u_n(\sin nh/nh)^k$ converges for all $h\neq 0$ in a neighbourhood of 0 and tends to S as $h\to 0$. For any trigonometrical series $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$, put, in particular,

$$R_h^k(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\left(\frac{\sin nh}{nh}\right)^k.$$

The series obtained from

$$-\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \log\left|2\sin\frac{x}{2}\right|$$

(x not a multiple of 2π) by repeated formal term-by-term differentiations are shown to present Gibbs phenomena. Let, namely, $\beta\geqq 0$ be any real number distinct from certain integers that depend only on the number of differentiations; and let further $x\to+0$ and $h\to 0$ such that $x/h\to\beta$. Then the following results hold. For the series $\sum_{n=1}^{\infty}\sin nx$ the limits of $R_h^1(x)(\frac{1}{2}\cot\frac{1}{2}x)^{-1}$ fill the interval $(0,\infty)$. For the series $\sum_{n=1}^{\infty} n\cos nx$ the limits of $R_h^2(x)(d[\frac{1}{2}\cot\frac{1}{2}x]/dx)^{-1}$ fill the same interval. For the series obtained by $k-1\geqq 2$ differentiations, the limits of $R_h^k(x)(d^{k-1}[\frac{1}{2}\cot\frac{1}{2}x]/dx^{k-1})^{-1}$ fill the whole interval $(-\infty,\infty)$. *K. Mahler* (Manchester)

6504:

Bhatt, Shri Nivas. An aspect of local property of $|R, \log n, 1|$ summability of Fourier series. Tôhoku Math. J. (2) **11** (1959), 13–19.

Let S_n denote the nth partial sum of $\sum a_n$ and $R_n=(\sum_1^n \nu^{-1}S_\nu)/\log n$. Then the series $\sum a_n$ is said to be absolutely summable $(R, \log n, 1)$, i.e., summable $|R, \log n, 1|$, if $\sum|R_n-R_{n+1}|<\infty$. For the Fourier series of an $L(0, 2\pi)$ integrable function

$$f(t) \sim \sum (a_n\cos nt + b_n\sin nt) = \sum An(t),$$

the author establishes that $|R, \log n, 1|$ summability of $\sum An(t)$ depends only on the behaviour of the function $f(t)$ in the immediate neighbourhood of $t=x$ if (*) $\sum|An(x)|(n\log n)^{-1}<\infty$, and this condition (*) is less stringent than the condition of Mohanty and Izumi [same J. (2) **8** (1956), 201–204; MR **20** #4142]. *M. Tomić* (Belgrade)

6505:

Bochner, S.; and Izumi, S. Strong law of large numbers for sequences of almost periodic functions. Ann. of Math. (2) **69** (1959), 718–732.

Soit $\{n_k\}$ une suite lacunaire satisfaisant à la condition d'Hadamard: $n_{k+1}/n_k\geqq q>1$. (1°) Soit f une fonction B^2-presque périodique, avec la valeur moyenne nulle, et soit $\sum(a_n\cos\lambda_n x+b_n\sin\lambda_n x)$ sa série de Fourier. Si la suite $\{\lambda_n\}$ n'a pas de point d'accumulation à l'origine, et si

$$\sum_{\lambda_n>M} (|a_n|^2+|b_n|^2)^{1/2} \leqq A/(\log\log M)^{1+\varepsilon} \quad (\varepsilon\geqq 0),$$

on a: $(1/N)\sum_{k=1}^{N} f(n_k x)\to 0$, sauf pour un ensemble de mesure moyenne nulle (c'est-à-dire sauf pour un ensemble E tel que, ω_E étant sa fonction caractéristique, on a $\lim_{T\to\infty}(1/2T)\int_{-T}^{T}\omega_E(x)dx=0$). (2°) Si $0<\lambda_n<1$,

$$f(x) = \sum(a_n\exp(2\pi i\lambda_n x)+a_{-n}\exp(-2\pi i\lambda_n x)),$$

$\sum(|a_n|+|a_{-n}|)/\lambda_n^{\varepsilon}<\infty$, et si $\omega(N)\to\infty$, on a:

$$|\textstyle\sum f(n_k x)| = o(N\cdot\log N\cdot\log\log N\cdot\omega(N))^{1/2},$$

p.p. (3°) Si $\lambda_n>1$, si f est S^2-presque périodique avec

$$f(x)\sim\sum(a_n\exp(i\lambda_n x)+a_{-n}\exp(-i\lambda_n x)),$$

$$\sum_{m}\Big(\sum_{m\leqq\lambda_n<m+1}(|a_n|+|a_{-n}|)\Big)^2<\infty,$$

(1) $$\sum_{\lambda_n\geqq M}(|a_n|^2+|a_{-n}|^2)\leqq A/(\log M)^{2\alpha} \quad (0<\alpha<1),$$

on a: (2) $(1/N^{1-\delta})\sum_{k=1}^{N} f(n_k x)\to 0$, p.p. pour $\delta<\alpha/2$. (4°) Si la condition (1) est remplacée par

$$\sum_{\lambda_n\geqq M}(|a_n|^2+|a_{-n}|^2)\leqq A/(\log\log M)^2,$$

on peut mettre dans (2), $\delta=0$. (5°) Si f est une fonction S^1-presque périodique avec la valeur moyenne nulle, on a pour tout $q>1$: $(1/N)\sum_{k=1}^{N} f(q^k x)\to 0$, p.p. *S. Mandelbrojt* (Paris)

6506:

Guinand, A. P. Concordance and the harmonic analysis of sequences. Acta Math. **101** (1959), 235–271.

It is not easy to abstract tangibly from this paper and we will only make a few remarks. Let $\{a_n, \alpha_n\}$ be a sequence of pairs in which α_n is real (a real non-positive) and a_n is complex. The author introduces a concept of almost periodicity for $\{a_n, \alpha_n\}$ by which it becomes reciprocally related to another such object $\{b_n, \beta_n\}$, such that, in many instances,

$$(2\pi)^{1/2}\lim_{T\to\infty}\frac{1}{2T}\sum_{|\beta_n|\leq T}' b_n e^{i\beta_n x}$$

has the value a_m for $z=\alpha_m$ and 0 elsewhere, and dually

$$(2\pi)^{1/2}\lim_{T\to\infty}\frac{1}{2T}\sum_{|\alpha_n|\le T}{}' a_n e^{-i\alpha_n z}$$

has the value b_m for $z=\beta_m$ and 0 elsewhere.

In certain cases

$$\sum_{n=-\infty}^{\infty} a_n G(\alpha_n) = \sum_{n=-\infty}^{\infty} b_n F(\beta_n),$$

where $F(x)$ is an arbitrary function, say in L, and $G(x)$ its Fourier transform. In other cases there are constants A, B, such that the two functions

$$x^{-1}\Big\{\sum_{0\le\alpha_n\le x}{}' a_n - Ax\Big\},\quad x^{-1}\Big\{\sum_{0\le\beta_n\le x}{}' b_n - Bx\Big\}$$

are complex Hankel transforms of a certain fixed type.

S. Bochner (Princeton, N.J.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 6494, 6495, 6951, 6953.

6507:

Pelczyński, A. On Mikusiński theorem of bounded moments. Acta Math. Sinica **9** (1959), 10–16. (Chinese and English)

The main theorem of this note is the following: For each continuous function $x(t)$ in $[1, b]$ $(b>1)$ such that $x(1)=0$, there is a sequence of polynomials $P_n(t)=\sum_0^{s_n} a_k^n t^k$ such that (1) $\lim P_n(t)=x(t)$ uniformly in $[1, b]$, and (2) $\lim \sum_1^{s_n} |a_k^n| = 0$.

From this theorem the following generalization of a theorem of Mikusiński [Colloq. Math. **2** (1951), 138–191; MR **13**, 214] is immediate. Let $g(t)$ be a function of bounded variation on $[1, b]$ vanishing and continuous at $t=1$ and such that there exists a constant M so that $|\int_1^b t^n dg(t)| \le M$ $(n=1, 2, \cdots)$. Then $g(t)=0$ on $[1, b]$.

The author also shows that the first theorem is a consequence of the second theorem by proving a theorem in general Banach spaces. If X is an arbitrary Banach space a sequence $\{x_n\}\subseteq X$ is called weakly divergent to infinity if, for each non-trivial linear functional f over X, $\sup |f(x_n)| = \infty$. The following theorem holds: The sequence $\{x_n\}$ is weakly divergent to infinity if and only if for each $\varepsilon>0$ and $x\in X$ there is a linear combination $\sum_1^n a_k x_k$ such that

$$\Big\|\sum_1^n a_k x_k - x\Big\| \le \varepsilon,\quad \sum_1^n |a_k| \le \varepsilon.$$

A. Devinatz (St. Louis, Mo.)

6508:

Ellis, H. W.; and Mott, T. E. On continuous linear transformations of integral type. Canad. Math. Bull. **2** (1959), 163–166.

The familiar theorem that any kernel k in $L^2(\mu\otimes\nu)$ induces a bounded linear transformation $K: L^2(\nu)\to L^2(\mu)$ with $\|K\| \le \|k\|_{L^2(\mu\otimes\nu)}$ is here generalized to a similar statement about kernels which induce maps between certain pairs of the Ellis-Halperin spaces L^λ. Two results about almost everywhere convergence are proved. [The L^λ spaces generalize the L^p spaces: see Canad. J. Math. **5** (1953), 576–592; MR **15**, 439.]

D. A. Edwards (Newcastle-upon-Tyne)

6509:

van der Corput, J. G. On a limit theorem in the theory of the Laplace transformation. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 1–4.

The author states the following theorem. Let $F(t)\ge 0$ for $t\ge 0$ and let $f(s)=\int_0^\infty e^{-st}F(t)dt$ converge for $\mathrm{Re}\,(s)>1$; for constants $A\ge 0$ and $a>0$, suppose that $f(s)-A/s-1$ converges to a limit function as $\mathrm{Re}(s)$ approaches 1, uniformly in an interval $-a\le \mathrm{Im}\,(s)\le a$; finally suppose that for every positive ε, there is a positive δ such that, for all x and x' with $x\le x'\le x+\delta$, $F(x)\le(1+\varepsilon)F(x')$. Then $\lim e^{-t}F(t)=A$. The result was established by G. Doetsch under the additional assumption that $F(t)$ is monotone decreasing. By using a certain lemma, the author shows that the monotonicity condition can be omitted. However, there seems to be an error in the proof of the lemma: In the proof of formula (3) on page 3, it is asserted that since an integral over an interval I is greater than $A-\varepsilon$, the same is true of the integral over an interval J. But $J=[-\varphi(\delta v), \varphi(\delta v)]$ is a subinterval of $I=[\alpha(u-\delta/2, v), \beta(u-\delta/2, v)]$ for u sufficiently large.

D. L. Bernstein (Towson, Md.)

6510:

Horváth, J. On some composition formulas. Proc. Amer. Math. Soc. **10** (1959), 433–437.

Let $e_1, \cdots, e_n$ be the canonical basis of R^n and $Q=\sum_{j=0}^\infty Q_j$ the commutative graded algebra generated by the elements $e_1, \cdots, e_n$, which satisfy the relation $e_1^2+\cdots+e_n^2=0$. Q_0 is identified with R and Q_1 with R^n. The author proved in a previous paper [Trans. Amer. Math. Soc. **82** (1956), 52–63; MR **19**, 270] that if $x=x_1e_1+\cdots+x_ne_n$ then the components of x^j with respect to a basis in Q_j form a maximal system of homogeneous harmonic polynomials $Y_j(x)$ of degree j. The nabla operator $\nabla=\sum_1^n e_i\partial/\partial x_i$ operates on distributions taking their values in Q, and ∇^j $(j=0, 1, 2, \cdots)$ is well defined. Let

$$R_\alpha = \Gamma(\tfrac{1}{2}(n-\alpha))[\pi^{n/2}2^\alpha\Gamma(\alpha/2)]^{-1}r^{\alpha-n},$$

$r=(x_1^2+\cdots+x_n^2)^{1/2}$, be the kernel of M. Riesz and let $K_{\alpha j}=(-\nabla)^j R_{\alpha+j}$. The author proves that

$$(1)\qquad K_{\alpha j} = \Gamma(\tfrac{1}{2}(n-\alpha+j))[\pi^{n/2}2^\alpha\Gamma(\tfrac{1}{2}(\alpha+j))]^{-1}x^j r^{\alpha-n-j},$$

$$(2)\qquad K_{\alpha j} * K_{\beta l} = K_{\alpha+\beta, j+l},$$

$$(3)\qquad \mathfrak{F}(K_{\alpha j}) = (-i)^j(2\pi)^{-n}\xi^j\rho^{-j-\alpha}.$$

For $\alpha=0$ these formulas reduce to the results of the author given in the mentioned paper. From (3) he derives the Bochner formula for $\mathfrak{F}(Y_j(x)r^{\alpha-n-j})$. Finally he proves that if $\kappa(\sigma)$ is a continuous function on the unit sphere and $K=\kappa(\sigma)/r^{n-\alpha}$, $0\le\alpha<n$, then $T\to K*T$ is a continuous map from $\mathcal{D}_{L^p}'$ into $\mathcal{D}_{L^q}'$, $1/q=1/s+1/p-1<1/p-\alpha/n$, $1<p<q<\infty$.

M. Cotlar (Buenos Aires)

6511:

Srivastava, K. J. Fractional integration and the $\tilde{\omega}_{\mu,\nu}$-transform. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 201–208.

Combinations of Hankel transforms in $L^2(0, \infty)$ are discussed. (It is known that if any "general transform" is

denoted by $Gf(t)$ and $Gf=t^{-1}f(t^{-1})$ by Rf then the combination G_1RG_2f is also a G; however, this basic fact is not used, but it is mentioned at the end of the paper.) With the aid of the Mellin transform, the fractional-integral operators

$$I_{\eta,\alpha}^{+}f = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt,$$

$$k_{\eta,\alpha}^{-}f = \frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty} \frac{(t-x)^{\alpha-1}}{t^{\eta+\alpha}}f(t)dt,$$

and a transform $\mathfrak{H}_{2\mu+2\nu,2\nu+2\alpha}f(t)$ that is derived from a combination of two Hankel transforms, formulae are deduced for $I_{\mu,\alpha}^{+}\{I_{\nu,\alpha}^{+}\}$ and $k_{\mu,\alpha}^{-}\{k_{\nu,\alpha}^{-}\}$, as well as the formula

$$I_{\mu,\alpha}^{+}\{I_{\nu,\alpha}^{+}\mathfrak{H}_{2\mu,2\nu}\} = \mathfrak{H}_{2\mu+2\alpha,2\nu+2\alpha}I_{\mu,\alpha}^{+}\{I_{\nu,\alpha}^{+}\}$$

and similar results. Finally, transformations of the form $G_1RG_2R\cdots G_n$ are considered, where the G_j are transformations of the Hankel type. *H. Kober* (Birmingham)

6512:

Gutmann, Marçian. Quelques opérations linéaires de type delta-Dirac. Bul. Inst. Politehn. Bucureşti **20** (1958), no. 2, 39–46. (Russian, English and German summaries)

In this paper the author gives some formulas, in symbolic form, for the derivatives of orderly functions (a function is orderly if it has only discontinuities of the first kind). Some formulas for the delta function are given and by means of them some results of Fourier, Bromwich and Kotelnikov are newly derived. The reviewer regrets that the author has left so many terms undefined, which makes the understanding of the paper difficult.

W. A. J. Luxemburg (Pasadena, Calif.)

6513:

Gutmann, Marcian. La convergence en delta et le calcul opérationnel singulier. Bul. Inst. Politehn. Bucureşti **20** (1958), no. 2, 47–54. (Russian, English and German summaries)

Let $f(t)$ be a function defined for all $t\geq 0$ and let f be continuous except at a countable number of points $\{t_k\}$, where the function becomes infinite of order n_k. If the integral $\int_0^{\infty} f(t)e^{-pt}dt$ diverges, then the author defines the Laplace transform of such a function to be $\int_0^{\infty} f_\delta(t)e^{-pt}dt$, where

$$f_\delta(t) = f(t) - \sum_{k=1}^{\infty}\sum_{j=0}^{n_k}\lambda_{jk}\delta^{(j)}(t-t_k)$$

is such that the last integral converges (here δ stands for the delta function). This method is illustrated by means of a number of examples. In the reviewer's opinion this method deserves a more rigorous treatment than the one given in this paper.

W. A. J. Luxemburg (Pasadena, Calif.)

6514:

Butzer, P. L. Singular integral equations of Volterra type and the finite part of divergent integrals. Arch. Rational Mech. Anal. **3**, 194–205 (1959).

The author applies Mikusiński's theory of convolution quotients [Mikusiński, Studia Math. **11** (1949), 41–70; **12** (1951), 80–83, 208–224, 227–270; *Rachunek operatorów*, Polskie Towarzystwo Matematyczne, Warsaw, 1953; MR **12**, 189; **13**, 131, 740, 751; **16**, 243] to a discussion of the finite part of a divergent integral of the form

$$(*)\qquad \int_0^t (t-u)^{-\alpha}f(u)du, \quad t>0,$$

and to integral equations of the first and second kinds involving such finite parts. He defines an operator corresponding to the function $t^{-\alpha}$, $\alpha\geq 1$, as follows. If $\alpha=k=1, 2, \cdots$, then $\{\mathrm{FP}\ t^{-k}\}=s^k h$, where

$$(k-1)!h(t) = (-1)^{k-1}\log t + \sum_{n=0}^{k-2}\frac{(-1)^{n+1}}{k-n-1}\binom{k-1}{n}$$

(an empty sum being interpreted as zero), and s is Mikusiński's operator of differentiation; and if $\alpha=k+\beta$ where $k=1, 2, \cdots$ and $0<\beta<1$, then

$$\{\mathrm{FP}\ t^{-k-\beta}\} = \frac{(-1)^k\pi s^{k+\beta-1}}{\Gamma(k+\beta)\sin\beta\pi}.$$

He shows that for $\alpha=k+\beta$, $0\leq\beta<1$, and a function f for which $d^{k-1}f/dt^{k-1}$ exists and is absolutely continuous for $t\geq 0$ and $f(0)=f'(0)=\cdots=f^{(k-2)}(0)=0$, the finite part of (*) can be represented as $\{\mathrm{FP}\ t^{-\alpha}\}\{f(t)\}$. This result is then applied to integral equations involving (*).

A. Erdélyi (Pasadena, Calif.)

6515:

Saxena, R. K. A theorem on Meijer transform. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 166–170.

If

$$\phi(p) = (2/\pi)^{\frac{1}{2}}p\int_0^{\infty}(pt)^{\frac{1}{2}}K_\nu(pt)f(t)dt$$

and $\phi(p)$ is the operational image (p times the Laplace transform) of $t^{\lambda-1}f(t^{-1})$, the author represents $\phi(p)$ as an integral involving ϕ. He applies the resulting formula to the evaluation of some infinite integrals.

A. Erdélyi (Pasadena, Calif.)

6516:

Griffith, James L. A note on the finite Fourier transform. J. Austral. Math. Soc. **1** (1959/61), part 1, 95–98.

If $f(x)\in L^2(B, A)$ and $f(x)\sim K(A-x)^p$ for $x\to A^-$ and $f(x)\sim M(x-B)^q$ for $x\to B^+$ with $\mathrm{Re}\ p>-1/2$ and $\mathrm{Re}\ q>-1/2$, then the transform

$$(*)\qquad F(z) = (2\pi)^{-1/2}\int_B^A e^{izx}f(x)dx$$

is asymptotically equivalent to $(2\pi)^{-1/2}M\Gamma(q+1)v^{-q-1}e^{-vB}$ for $z=iv$ and $v\to+\infty$ and to $(2\pi)^{-1/2}K\Gamma(p+1)(-v)^{-p-1}e^{-vA}$ for $z=iv$ and $v\to-\infty$. Also, if these asymptotic conditions are satisfied by an integral function $F(z)$ of exponential type belonging to $L^2(-\infty, \infty)$ on the real axis, then there exists a function $f(x)$, not vanishing almost everywhere in any neighborhood of B or any neighborhood of A and such that (*) holds. *A. G. Azpeitia* (Amherst, Mass.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 6496, 6514, 6537, 6708.

6517:

Butlewski, Z. Sur la limitation des solutions d'un système d'équations intégrales de Volterra. Ann. Polon. Math. **6** (1959), 253–257.

Extending a result due to T. Sato [Tôhoku Math. J. **4** (1952), 272–274; MR **14**, 878] it is shown that if in the system of Volterra integral equations

$$u_j(x) = \sum_1^n \int_a^x a_{jk}(x,t)u_k(t)dt + b_j(x)$$

we have: (1) $a_{jk}(x,t)$ continuous on $a \leqq t \leqq x < \infty$, with $\max_{j,k} |a_{jk}(x,t)| \leqq A(x,t)$; (2) $b_j(x)$ bounded and continuous on $a \leqq x < \infty$, and $\sum_j |b_j(x)| \leqq B(x)$; and (3) $A(x,t)$ nonincreasing in t for each x; then

$$\sum_1^n |u_j(x)| \leqq \beta(x) \exp\left(n \int_a^x A(t,t)dt\right),$$

where $\beta(x) = \max_{a \leqq t \leqq x} B(t)$. This leads to conditions under which the solutions $u_j(x)$ are bounded as $x \to \infty$. A similar result holds if $A(x,t)$ is nondecreasing in t.

T. H. Hildebrandt (Chapel Hill, N.C.)

6518:

Grinberg, G. A. Integral equations in which the kernel depends on the absolute value of the difference between the arguments and the interval of variation of the variables is finite. Dokl. Akad. Nauk SSR **128** (1959), 450–453. (Russian)

The solution of an integral equation of the type described in the title is reduced to the solution of a pair of simpler integral equations.

H. P. Thielman (Ames, Iowa)

6519:

Schaefer, Helmut. Über algebraische Integralgleichungen mit nichtnegativen Koeffizienten. Math. Ann. **137** (1959), 385–391.

The paper deals with non-linear integral equations of the form

$$(1) \qquad \kappa^n y^n(s) - \sum_{\beta=0}^{n-1} \kappa^\beta y^\beta(s) a_\beta(s,y) = f(s).$$

Here κ is a real parameter, and s denotes a point varying over a compact subset Ω of the k-dimensional Euclidean space, while for $\beta = 0, 1, \cdots, n-1$

$$(2) \quad a_\beta(s,y) = \sum_{\alpha_1+\cdots+\alpha_\nu=n-\beta} \int_\Omega \cdots \int_\Omega K_{\beta\alpha_1\cdots\alpha_\nu}(s,t_1,\cdots,t_\nu) \times y^{\alpha_1}(t_1)\cdots y^{\alpha_\nu}(t_\nu)dt_1 \cdots dt_\nu.$$

The right member of (1) as well as the kernels in (2) are given functions on Ω which are continuous in all the arguments and, in addition, are supposed to be non-negative.

If in (1) $a_\beta(s,y)$ is replaced by $a_\beta(s,w)$, where w is an arbitrary continuous non-negative function of s, then the algebraic equation for y thus obtained is shown to have for $\kappa > 0$ a greatest non-negative root $F(\kappa,w,f)$. Then a sequence $\{y_m\}$ is defined by setting $y_0 = 0$, $y_{m+1} = F(\kappa, y_m, f)$. It is proved that there exists a non-negative κ_0 such that, for each κ in the open interval J defined by $\kappa > \kappa_0$, the sequence $\{y_m\}$ converges, and that the limit function $y^* = R_\kappa(f)$ is a solution of (1). If K is the cone of all non-negative continuous functions $\varphi(s)$ defined on Ω with the topology induced by $\|\varphi\| = \sup_{s\in\Omega} |\varphi(s)|$, then the map $R_\kappa(f)$ mapping $(\kappa, f) \in J \times K$ into $y^* \in K$ is lower semi-continuous; moreover it is monotone, i.e., in obvious notation, $y_1^* \leqq y_2^*$ if $f_1 \leqq f_2$ and $\kappa_1 \leqq \kappa_2$; in addition, there is a decreasing function $C = C(\kappa)$ which is continuous from the right such that $\|R_\kappa(f)\| \leqq C(\kappa)\|f\|^{1/n}$. If κ_0 is positive then κ_0 is an eigenvalue of (1) with an eigenfunction in K.

Finally it is shown that (1) has a positive eigenvalue κ_0 with a positive eigenfunction if for at least one $\beta = \beta_0$ $(0 \leqq \beta_0 \leqq n-1)$ one of the kernels $K_{\beta_0\alpha_1\cdots\alpha_\nu}$ is positive, and the following estimate for κ_0 is proved

$$\kappa_0 \geqq (\gamma^{1/\nu} m\Omega)^{\nu/(n-\beta_0)},$$

where γ is a positive lower bound for $K_{\beta_0\alpha_1\cdots\alpha_\nu}$ on Ω, and where $m\Omega$ denotes the measure of Ω.

The statement of the preceding paragraph is a direct generalization of a classical theorem of Jentzsch concerning linear Fredholm integral equations with positive kernel. Algebraic integral equations of the form (1) have been first treated by W. Schmeidler [Math. Nachr. **10** (1953), 247–255; Ann. Acad. Sci. Fenn. Ser. A. I. no. 220 (1956); MR **15**, 434; **18**, 302]. The relation of the results of the present paper to those of Schmeidler is discussed by the author. *E. H. Rothe* (Ann Arbor, Mich.)

6520:

Guglielmino, Francesco. Sulla risoluzione del problema di Darboux per l'equazione $s = f(x,y,z)$. Boll. Un. Mat. Ital. (3) **13** (1958), 308–318. (English summary)

In this paper the integral equation

$$(D) \quad z(x,y) = \sigma(x) + \tau(y) + \int_0^x \int_0^y f[\xi, \eta, z(\xi,\eta)]d\xi d\eta - z_0$$

in the unknown function $z(x,y)$ is considered, where $\sigma(x)$ and $\tau(y)$ are continuous on $I = [0,a]$ and $J = [0,b]$, respectively, and $\sigma(0) = \tau(0) = z_0$, and where $f(x,y,z)$ is defined on $S = I \times J \times E^1$, continuous in z for each (x,y) in $R = I \times J$ and measurable in R for every $z \in E^1$ (the entire real line). This equation is a generalization, in integral form, of the classical Darboux problem for the equation $z_{xy} = f(x,y,z)$. The following theorem is proved. There exists at least one solution of equation (D) provided that f satisfies the additional hypothesis that $|f(x,y,z)| \leqq M(x,y,|z|)$, where M is defined and non-negative in $S_0 = R \times E_+$, and is non-decreasing in the third variable u, for u in E_+ (the set of all non-negative u), and provided that there exists a function $\bar{u}$, non-negative in R, for which

$$\bar{u}(x,y) \geqq |\sigma(x) + \tau(y) - z_0| + \int_0 \int_0 M(\xi,\eta,\bar{u}(\xi,\eta))d\xi d\eta.$$

The method of proof consists in constructing, in an ingenious manner, a sequence $z^{(n)}$ of functions, which is equibounded and equicontinuous, so that a subsequence can be extracted which converges uniformly to a function $\bar{z}$ which is then shown to be a solution of (D). An interesting example shows that for the usual sequence of approximating functions z_n, defined by letting $z_0 = \sigma(x) + \tau(y) - z_0$, and then substituting successively in the right side of (D) to get $z_1, z_2, \cdots$, it is not necessarily true that the sequence converges to a solution of the equation. However, the author does give, in a second theorem, additional hypotheses on f which insure a unique solution of (D) and one that is the limit of the sequence z_n. This condition essentially, is that $|f(x,y,z) - f(x,y,z')| \leqq \psi(x,y,|z-z'|)$, where $\psi(x,y,u)$ is a non-negative function defined in S_0, continuous and non-decreasing in u, and for each u, summable in R and such that the integral equation $u(x,y) = \iint_T \psi(\xi,\eta,u(\xi,\eta))d\xi d\eta$ has only the solution $u = 0$ in

every subrectangle $T=I_a\times J$, where $I_a=[0, a]$ and $0<a\leq a$. (This condition is of course the generalization of a similar condition for ordinary differential equations.)

D. L. Bernstein (Towson, Md.)

6521:

Nersesyan, A. B. Expansion in eigenfunctions of an integro-differential equation with lagging argument. Dokl. Akad. Nauk SSSR **129** (1959), 511–514. (Russian)

The author considers a boundary value problem (L) for an integro-differential equation involving only second derivatives with respect to the dependent variable and having lagging arguments. By constructing a related boundary value problem (L*) with leading arguments whose eigenvalues are the same as those of (L), the author constructs a biorthogonal system from the eigenfunctions of (L) and (L*). An expansion theorem is then given in terms of this biorthogonal system. If (L) is specialized to a boundary value problem for a second order differential system then (L) is self-adjoint and the expansion theorem coincides with well known results.

J. K. Hale (Baltimore, Md.)

6522:

Lihtarnikov, L. M.; and Myakišev, V. P. Solution of a class of integro-differential equations involving partial derivatives by the Fourier method. Dokl. Akad. Nauk SSSR **127** (1959), 516–519. (Russian)

The authors consider the integro-differential equation

$$L[u] = f(x, t)+\lambda \iint_{[D]} k(x, t; z, \sigma)M[u]dzd\sigma,$$

where

$$L[u] = A(t)\frac{\partial^2 u}{\partial t^2}+B(t)\frac{\partial u}{\partial t}+C(x)\frac{\partial^2 u}{\partial x^2}$$
$$+D(x)\frac{\partial u}{\partial x}+[E_1(t)+E_2(x)]u(x, t),$$

$$M[u] = a(z, \sigma)\frac{\partial u}{\partial z}+b(z, \sigma)\frac{\partial u}{\partial \sigma}+c(z, \sigma)u(z, \sigma),$$

and $[D]$ is the rectangle $a\leq x\leq b$, $0\leq t\leq t_0$, subject to initial and boundary conditions

$$u(x, 0) = \varphi(x);\quad u_t'(x, 0) = \psi(x),$$

$$a_0u(a, t)+b_0u_x'(a, t) = 0\quad a_1u(b, t)+b_1u_x'(b, t) = 0,$$

where a_0, b_0, a_1 and b_1 are constants.

Under suitable continuity restrictions and assumptions regarding the signs of the prescribed functions, the authors expand various quantities in terms of eigenfunctions of

$$C(x)X_n''(x)+D(x)X_n'(x)+E_2(x)X_n(x)+\mu_nX_n(x) = 0,$$

transform each term in the series and, summing up again, reduce the solution of the equation (1) to the solution of a standard second kind Fredholm type integral equation.

B. Davison (Toronto, Ont.)

6523:

Alihanova, R. I. A problem for a functional equation. Dokl. Akad. Nauk Azerbaĭdžan. SSR **15** (1959), 371–374. (Azerbaijani. Russian summary)

The integro-differential equation

$$\frac{\partial u}{\partial t} = \sum \phi^{(m_1,\cdots,m_n)}\left[\int_0^l\cdots\int_0^l u^2dx_1\cdots dx_n\right]\frac{\partial^{m_1+\cdots+m_n}u}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}$$

is studied, the summation being taken for all sets of positive integers $m_1, \cdots, m_n$ with $\sum m_i\leq 2p$, under the conditions $u|_{t=0}=F_1(x_1, \cdots, x_n)$ and $u|_L=0$, where L is the boundary of the set $0\leq x_s\leq l$, $s=1, 2, \cdots, n$. The function $\phi^{(m_1,\cdots,m_n)}(z)$ is supposed continuous for all $z>0$ and $F_1(x_1, \cdots, x_n)$ expandable in a Fourier sine-series. A solution is sought in the form

$$u(t, x_1, \cdots, x_n) = \sum A_{k_1,\cdots,k_n}(t)\sin k_1x_1\cdots\sin k_nx_n.$$

The problem is reduced to the solution of an infinite system of ordinary differential equations

$$A_{k_1,\cdots,k_n}'(t)+\sum \phi^{(m_1,\cdots,m_n)}\left[\left(\frac{\pi}{2}\right)^n\sum A_{i_1,\cdots,i_n}(t)^2\right]$$
$$\times(k_1)^{m_1}\cdots(k_n)^{m_n}A_{k_1,\cdots,k_n}(t) = 0,$$

the summation for the m's as above, with the initial conditions $A_{k_1,\cdots,k_n}(0)=a_{k_1,\cdots,k_n}$, and this system is solved by a reduction method. The existence of solutions for both problems is proved, as are certain lemmas.

Author's summary

FUNCTIONAL ANALYSIS

See also 6400, 6472, 6473, 6507, 6967.

6524:

★Cristescu, Romulus. Spaţii liniare ordonate. [Ordered linear spaces.] Biblioteca Matematică, III. Editura Academiei Republicii Populare Romîne, Bucharest, 1959. 330 pp. Lei 11.10.

This book contains 9 chapters whose subjects are the following. I: Ordered sets (preliminaries on order, lattices, boolean lattices, limits). II: Ordered linear spaces and their properties (ordered, reticulated, completely reticulated spaces). III: Special types of ordered linear spaces. IV: Concrete spaces (euclidean, of real functions, of measurable functions, of functions of bounded variation, of analytic functions). V: Linear and bilinear operators (positive operators and their difference, continuity, sets of functionals, disjunctive operators, bilinear operators). VI: Operator-products and ordered algebras. VII: Integrals in ordered linear spaces (integral representations of linear operators). VIII: The ergodic theorem (individual and in the mean). IX: Equations in ordered linear spaces (linear and non-linear; special spaces). There is a bibliography with 206 titles, including some 45 in Russian. The discussion is detailed. Each chapter is followed by some pages of comments of a historical nature.

E. R. Lorch (New York, N.Y.)

6525:

★Marinescu, G. Spaţii vectoriale topologice şi pseudo-topologice. [Topological and pseudo-topological vector spaces.] Biblioteca Matematică, IV. Editura Academiei Republicii Populare Romîne, Bucharest, 1959. 217 pp. Lei 8.80.

This book surveys the subject of its title and some of the applications which have been made. To save space, all

proofs which are direct adaptations of the corresponding proofs for normed spaces are omitted and reference is made to the author's earlier book [*Spații vectoriale normate*, Editura Acad. R. P. R., Bucharest, 1956; MR **19**, 44].

After preliminary pages on convergence, ordered sets, linear spaces, and linear topological spaces the author comes to the main theme of the book, the pseudo-topological and polynormed spaces, which describe structures naturally available in conjugate spaces and more general spaces of linear operators. These are exploited in the second chapter, on spaces of linear operators, in a discussion of the relations between the topologies of uniform convergence in the space of linear operators from E into F. This chapter also discusses duality and certain special spaces, such as Montel, tonnelé, or bornological spaces, and concludes with a section on completely continuous operators.

The remaining chapters are brief introductions to a number of topics in this general setting. Chapter headings are: III. Tensor product spaces. IV. Differential and integral calculus. V. Finitely and countably additive, vector-valued set-functions. VI. Algebras. VII. Differential geometry. VIII. Degree and fixed points of a mapping. IX. Special spaces and applications.

For those who have the author's earlier book at hand, and who are not dismayed by a presumed lack of knowledge of the Romanian language, this is a clear introduction to most of the topics which have been studied in linear topological spaces. *M. M. Day* (Urbana, Ill.)

6526:

Raĭkov, D. A. On two classes of locally convex spaces which are important in applications. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 22–34. (Russian)

A further extension of the theory of (LN^*) and (M^*) spaces [cf. Sebastião e Silva, Rend. Mat. e Appl. (5) **14** (1955), 388–410; MR **16**, 1122]. Here "convex space" = "locally convex topological linear space". Two increasing sequences of convex spaces $E_1 \subset E_2 \subset \cdots$ and $F_1 \subset F_2 \subset \cdots$ are said to be equivalent if, for each m, there exist indices p and q such that $E_m \subset F_p$ and $F_m \subset E_q$ and that the inclusion mappings $E_m \to F_p$ and $F_m \to E_q$ are continuous. The inductive limits of both sequences are then identical. If E and F are two Hausdorff convex spaces, a linear mapping f of E into F is said to be compact [totally continuous] if there exists a neighbourhood of zero U such that $f(U)$ is compact [the closure of $f(U)$ is compact]. A sequence of convex spaces is said to be regular [strongly regular] if $E_n \subset E_{n+1}$ and the inclusion mapping $E_n \to E_{n+1}$ is totally continuous [compact]. Theorem 1: Every regular sequence of convex spaces is equivalent to a strongly regular sequence of Banach spaces.

A sequence $K_1, K_2, \cdots$ of compact subsets of a convex space E is said to be regular if (1) the K_i are absolutely convex, (2) each K_n is contained in the algebraic interior of K_{n+1}, (3) E is the union of all K_i, (4) each K_n is compact in the Banach space spanned by K_{n+1} with K_{n+1} as unit ball. A topological space M is said to be the free union of subspaces M_α if $M = \bigcup M_\alpha$ and if a set F is closed if and only if $F \cap M_\alpha$ is closed in M_α for each α. Theorem 2: Each inductive limit of a regular sequence of convex spaces contains a regular sequence of compact sets. An inductive limit E of a regular sequence of convex spaces is the free union of each regular sequence of compact subsets of E. Most of the known properties of (LN^*) spaces are shown to be immediate consequences of theorem 2. In a similar manner, the following result follows easily: Each factor space and each closed subspace of an (LN^*) space is again an (LN^*) space. By duality [cf. Sebastião e Silva, loc. cit., prop. 7] this result holds for (M^*) spaces as well. Another consequence of theorem 2 is the following open mapping theorem: Each continuous linear mapping of an (LN^*) space onto a (c) space is open. Here, a (c) space is a convex space which does not admit a finer convex topology with the same family of absolutely convex compact sets. *V. Ptàk* (Prague)

6527:

Plans, Antonio. Zerlegung von Folgen im Hilbertraum in Heterogonalsysteme. Arch. Math. **10** (1959), 304–306.

A heterogonal system in a separable Hilbert space H is a sequence of vectors $A_1, A_2, \cdots$ satisfying:

$$0 < m < |A_n| < M < \infty;$$
$$\alpha([A_{p_1}, \cdots, A_{p_n}], [A_{q_1}, \cdots, A_{q_n}]) > \lambda > 0, \quad p_i \neq q_j.$$

Here $\alpha(V_1, V_2)$ denotes the angle between the two subspaces V_1 and V_2. The author proves the following theorem: Let $S = \{A_n\}$ satisfy $0 < m < |A_n|$, $A_n \in H$, and $A_n \not\to 0$ (weak convergence). Then $S = \bigcup S_i$ ($i = 1, \cdots, r$ or $i = 1, 2, \cdots$) where S_i is a heterogonal system. The corresponding angles λ_i may be so chosen that $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$, $\lambda_i < \pi/2$, and $\lambda_i \to \pi/2$ if $r = \infty$. The proof is based on the following lemma: If $0 < m < |X_n| < M < \infty$, then $X_n \not\to 0$ if and only if $\alpha(X_n, V) \to \pi/2$ for every finite-dimensional subspace V. *E. R. Lorch* (New York, N.Y.)

6528:

Pełczyński, A.; and Semadeni, Z. Spaces of continuous functions. III. Spaces $C(\Omega)$ for Ω without perfect subsets. Studia Math. **18** (1959), 211–222.

[For part II, see Studia Math. **16** (1957), 193–199; MR **19**, 1184. See also #6571.] A topological space is said to be dispersed if it contains no non-void perfect subset. The main theorem of the paper gives 16 necessary and sufficient conditions for a compact Hausdorff space Ω to be dispersed. Each condition is in terms of one of the following: Ω itself, the Banach space $C(\Omega)$ of real-valued continuous functions on Ω, the first or second conjugate space of $C(\Omega)$, or the Borel measures on Ω. Several conditions involve the concept of linear dimension, the most important one being: every infinite-dimensional subspace of $C(\Omega)$ contains isomorphically the space c of convergent sequences. Some of the implications in the main theorem were obtained earlier by W. Rudin [Proc. Amer. Math. Soc. **8** (1957), 39–42; MR **19**, 46].

The authors state that the spaces $C(\Omega)$ with Ω dispersed enable one to construct counter-examples to many general statements concerning Banach spaces. As an illustration, they deduce from the main theorem that there are many examples of non-isomorphic Banach spaces $C(\Omega)$ with equivalent conjugate spaces, even for metrizable Ω, thus providing a solution to a problem of Banach [*Théorie des opérations linéaires*, Monogr. Mat. 1, Warsaw, 1932, p. 243]. *C. W. Kohls* (Rochester, N.Y.)

6529:

Schäffer, Juan Jorge. Addendum: Function spaces with translations. Math. Ann. **138** (1959), 141–144.

Some results concerning local integrability conditions (and the sharpness of the bounds in conditions of this kind) are added to the results in an earlier paper of the author [Math. Ann. **137** (1959), 209–262; MR **21** #287].

A. C. Zaanen (Leiden)

6530:

Ellis, H. W. On generalized Morse-Transue function spaces. Canad. J. Math. **11** (1959), 416–426.

This paper continues the study of MT*-spaces introduced by the author [same J. **10** (1958), 381–391; MR **20** #1901] and generalizing the MT-spaces defined by Marston Morse and the reviewer [J. Analyse Math. **4** (1955), 149–186; MR **17**, 469]. In the study of MT-spaces a central role was played by the fact that the semi-norm of an element x of an MT-space A is given by

$$\mathfrak{N}^A(x) = \sup_{\eta} \int^* |x|d|\eta|, \tag{1}$$

where the sup is taken over all sub-unit measures of the measure dual (isomorphic with the topological dual). Formula (1) need not hold in an MT*-space, with the measure dual replaced by the MT-measure conjugate $\mathfrak{N}^*$ (the MT*- measure conjugate of MR **20** #1901), nor can one prove as in the case of MT-spaces that, if η is in $\mathfrak{N}^*$, $|\eta|_\varepsilon$ is also and has the same norm (with $|\eta|_\varepsilon$ defined as in the second reference above). The author is thus led to define an $\mathfrak{N}^A$-extensible MT*-space as an MT*-space which possesses these two properties, and shows that such spaces have many of the properties of MT-spaces.

There follows an investigation of the relations between $\mathfrak{N}^A$-extensible MT*-spaces and λ-spaces, the function spaces determined by a length function λ and dependent on a fixed measure μ, introduced by the author and I. Halperin [Canad. J. Math. **5** (1953), 576–592; MR **15**, 439]. When the underlying space E is countable at infinity, the results can be simply stated as follows. If A is a λ-space containing the continuous functions with compact support, A is an $\mathfrak{N}^A$-extensible MT*-space for which every measure in $\mathfrak{N}^*$ is of base μ. Conversely, if A is an $\mathfrak{N}^A$-extensible MT*-space for which every measure in $\mathfrak{N}^*$ is of base μ, $\mathfrak{N}^A$ extended by (1) determines a length function λ, and the λ-space thereby determined coincides with Ω^A on some μ-measurable set B such that $E-B$ is $\mathfrak{N}^*$-negligible, where Ω^A is the space of functions integrable with respect to every measure in $\mathfrak{N}^*$.

W. R. Transue (Gambier, Ohio)

6531:

Bouix, Maurice. Les distributions d'une variable: relations entre deux définitions, dérivation, intégration. Ann. Télécommun. **14** (1959), 236–255.

L'étude du champ électromagnétique émis par une source harmonique en présence d'obstacles introduit la recherche de certaines solutions du système de Maxwell et de l'équation vectorielle d'Helmholtz. On introduit d'abord d'après J. Mikusiński et R. Sikorski les distributions d'une variable comme limites d'une suite de fonctions, et on en déduit l'existence d'une dérivée; on étudie les séries de distributions, leurs primitives; l'introduction de la notion de valeur d'une distribution en un point permet de préciser qu'une distribution, tout au moins d'ordre fini, résulte de la superposition d'une fonction continue ou discontinue et d'un ensemble de distributions de Dirac ou de leurs dérivées. On expose ensuite comment la notion de distribution résulte de celle de fonctionnelle linéaire définie sur un espace fonctionnel d'après les travaux de L. Schwartz et l'on raccorde cette deuxième définition à la première. En conclusion on remarque que les propriétés des distributions s'introduisent mieux par le premier procédé, mais que le deuxième procédé est souvent plus commode pour les calculs.

Résumé de l'auteur

6532:

Yoshinaga, Kyôichi; and Ogata, Hayao. On convolutions. J. Sci. Hiroshima Univ. Ser. A **22** (1958/59), 15–24.

The usual notation of the theory of distributions of Schwartz is used. Let S, T be distributions such that $(S*\phi)(x)(\check{T}*\psi)(x) \in L$ for any $\phi, \psi \in D$; then we say (following Chevalley) that S and T are composable and we set $(S*T, \phi*\psi) = \int (S*\phi)(x)(T*\psi)(x)dx$, it being shown that this formula defines $S*\check{T} \in D'$. For any subset $A \subset D'$ we define A^*, the c-dual of A, as the set of all distributions composable with the distributions of A. Then the author determines the c-duals of the spaces considered by Schwartz.

L. Ehrenpreis (New York, N.Y.)

6533:

Yoshinaga, Kyôichi. On the space $(\mathscr{S}')$ and the Fourier transformation. Bull. Kyushu Inst. Tech. Math. Nat. Sci. **5** (1959), 1–11.

The Schwartz space $\mathscr{S}'$ of slowly increasing distributions is identified with a canonical inductive limit of spaces $\mathfrak{N}_p$ of continuous functions on R^n which are of order $|x|^{2np}$ at infinity, and the Fourier transform of elements of $\mathscr{S}'$ is defined explicitly by a method which, as far as formalism goes, is essentially a case of Bochner's method of defining generalised Fourier transforms, but is supplemented by a discussion of the continuity of the operations involved.

J. L. B. Cooper (Cardiff)

6534:

Schwartz, Laurent. Théorie des distributions à valeurs vectorielles. I. Ann. Inst. Fourier, Grenoble **7** (1957), 1–141.

Cet article n'est que le 1er chapitre d'un livre sur la théorie des distributions vectorielles, que l'A. avait esquissée dans son Séminaire 1953–54 [Secrétariat Mathématique, Paris, 1954; MR **17**, 764].

La théorie est essentiellement basée sur la notion de produit ε d'espaces localement convexes $L_1, \cdots, L_n$, étudiée au § 1. On appelle produit ε des L_i l'espace vectoriel des formes multilinéaires sur le produit des espaces $(L_i)_{\sigma}'$, hypocontinues par rapport aux parties équicontinues des L_i', muni de la topologie de la convergence uniforme sur les produits de ces parties. On désigne ce produit par $\varepsilon_1^n L_i$, ou par $L_1 \varepsilon L_2$ si $n=2$ (L_σ' représente, en général, le dual de l'espace L muni de la topologie de la convergence compacte). On peut considérer, plus généralement, l'espace analogue $\varepsilon_1^n(L_i; M)$ d'applications multilinéaires du produit des $(L_i)_\sigma$ dans un espace M, mais on démontre que $\varepsilon_1^n(L_i; M)$ s'identifie au produit ε de $\varepsilon_1^n L_i$ par M; d'ailleurs, le produit ε est associatif. Beaucoup d'autres propriétés importantes de ce produit sont

établies au § 1; nous signalerons ici les deux faits suivants: l'espace $\varepsilon_1^n L_l$ est toujours quasi-complet, et complet si les L_l sont complets; le produit tensoriel topologique $L \otimes_\varepsilon M$, dont le complété est noté $L \hat{\otimes} M$ par Grothendieck dans sa thèse [Mem. Amer. Math. Soc. no. 16 (1955); MR 17, 763], est un sous-espace vectoriel topologique de $L\varepsilon M$, où il est dense pour tout M, si et seulement si L vérifie la "propriété d'approximation".

Le § 2 est consacré à la définition et à l'étude de plusieurs types généraux de distributions vectorielles. On appelle distribution d'ordre $\leqq m$ $(0 \leqq m \leqq \infty)$ sur R^n, à valeurs dans un espace localement convexe E (séparé), toute application linéaire continue T de $\mathscr{D}^m$ dans E, où $\mathscr{D}^m$ est l'espace des fonctions m fois continûment différentiables à support compact dans R^n. L'espace de ces distributions, c.-à.-d. l'espace $\mathscr{L}(\mathscr{D}^m; E)$, est noté $\mathscr{D}'^m(E)$ ou simplement $\mathscr{D}'(E)$ dans le cas où $m = \infty$; on le munit en général de la topologie de la convergence compacte, qui coïncide avec celle de la convergence bornée pour m infini. On démontre que $\mathscr{D}_c'^m(E)$ est canoniquement isomorphe à $\mathscr{D}_c'^m \varepsilon E$; il en résulte que, si E est complet, $\mathscr{D}_c'^m(E)$ s'identifie à $\mathscr{D}_c'^m \hat{\otimes}_\varepsilon E$. D'autre part, si $\mathscr{H}$ est un espace de distributions (sous-espace de l'espace $\mathscr{D}'$ des distributions scalaires sur R^n, muni d'une topologie l.c. plus fine que celle induite par $\mathscr{D}'$), on pose $\mathscr{H}(E) = \mathscr{H} \varepsilon E$ et on voit que $\mathscr{H}(E)$ est un sous-espace de $\mathscr{D}'(E)$, muni d'une topologie plus fine que la topologie induite. On dit que $\mathscr{H}$ a la propriété (ε) si, quel que soit l'espace quasi-complet E, le produit scalaire $\langle T, e' \rangle$ se trouve dans $\mathscr{H}$ pour toute $T \in \mathscr{H}(E)$ et tout $e' \in E'$ (on pose par définition $\langle T, e' \rangle(\varphi) = \langle e', T(\varphi) \rangle$). La plupart des espaces $\mathscr{H}$ rencontrés dans la pratique vérifient la propriété (ε).

Dans le § 3 on examine des exemples concrets de distributions vectorielles et on fait une première étude des opérations fondamentales sur ces distributions: dérivation, multiplication, convolution et transformations de Fourier et de Laplace (l'étude, très délicate, du produit scalaire, multiplicatif ou convolutif de deux distributions vectorielles est approfondie dans le chapitre II). Soit f une fonction à valeurs dans E, telle que, pour tout $e' \in E'$, le produit $\langle f, e' \rangle$ soit une fonction (scalaire) localement sommable dans R^n; alors, sous certaines conditions, il existe une distribution unique $T \in \mathscr{D}'(E)$ telle que $\langle T, e' \rangle = \langle f, e' \rangle$ pour tout e', et on identifie f à T. La dérivation peut être définie comme dans le cas des distributions scalaires. Quant à la multiplication, on considère deux espaces $\mathscr{H}$ et $\mathscr{L}$ de distributions, $\mathscr{H}$ étant "normal"; sous certaines restrictions, on définit le produit $ST \in \mathscr{L}(E)$ de $S \in \mathscr{D}'$ par $T \in \mathscr{H}(E)$, en posant $\langle ST, e' \rangle = S\langle T, e' \rangle$ pour tout $e' \in E'$. Définition analogue pour la convolution. La transformée de Fourier $\mathscr{F}T$ d'une distribution $T \in \mathscr{S}'(E)$ est définie encore en posant $(\mathscr{F}T)(\varphi) = T(\mathscr{F}\varphi)$ pour toute $\varphi \in \mathscr{S}$. Ensuite l'A. généralise, au cas des distributions vectorielles, sa théorie de la transformation de Laplace; si E est un espace $(\mathscr{DF})$, au sens de Grothendieck [Summa Brasil. Math. 3 (1954), 57–123; MR 17, 765], l'image de Laplace d'une distribution convenable T à valeurs dans E est une fonction holomorphe à valeurs dans E au sens usuel; mais, dans le cas général, l'image de Laplace de T n'est qu'une fonction holomorphe au sens large, d'après Grothendieck [J. Reine Angew. Math. 192 (1953), 77–95; MR 15, 963].

À la fin du § 3 on étudie la structure des distributions $T \in \mathscr{D}'(E)$ comme sommes de dérivées de mesures ou de fonctions continues. On sait que, si E coïncide avec le corps des scalaires, toute distribution est localement d'ordre fini; cela reste encore vrai si E est un espace $(\mathscr{DF})$ quasi-complet; mais, dans le cas général, on se trouve dans une situation analogue à celle des fonctions holomorphes au sens large de Grothendieck. {Remarque: ce fait a permis au rapporteur de construire une théorie directe des distributions vectorielles en termes de dérivées de fonctions vectorielles continues ["Sur la définition et la structure des distributions vectorielles", Publ. Centro Est. Mat. de Lisboa, 1958; Portugal. Math. 19 (1960), 1–80].}

Le § 4 est consacré à l'étude des noyaux distributions, basée sur les produits tensoriels topologiques d'espaces de distributions. Soient X^l, X^m des espaces euclidiens; un "noyau" est un élément de $\mathscr{D}_{x,y}'$, c'est-à-dire une distribution scalaire $T_{x,y}$ sur $X^l \times X^m$. L'A. présente d'abord son "théorème des noyaux" sous la forme: $\mathscr{D}_{x,y}' = \mathscr{D}_x'(\mathscr{D}_y') = \mathscr{D}_x' \hat{\otimes}_\varepsilon \mathscr{D}_y'$. Il considère ensuite les espaces de noyaux de la forme $\mathscr{H}_x \varepsilon \mathscr{K}_y$, où $\mathscr{H}_x$ et $\mathscr{K}_y$ sont des espaces de distributions $(x \in X^l, y \in Y^m)$ et il donne des critères d'appartenance à ces espaces. Les espaces $\mathscr{H}$ et $\mathscr{K}$ peuvent être de même nature: on trouve alors des identités telles que $\mathscr{E}_{x,y} = \mathscr{E}_x \hat{\otimes} \mathscr{E}_y$, $\mathscr{S}_{x,y} = \mathscr{S}_x \hat{\otimes} \mathscr{S}_y$, $\mathscr{E}_{x,y}' = \mathscr{E}_x' \hat{\otimes} \mathscr{E}_y'$, etc. Mais il y a aussi des cas importants où $\mathscr{H}$ et $\mathscr{K}$ ne sont pas de même nature: c'est le cas de l'espace $\mathscr{E}_x(\mathscr{D}_y') = \mathscr{E}_x \hat{\otimes} \mathscr{D}_y'$ des "noyaux semi-réguliers en x", de l'espace $\mathscr{E}_x'(\mathscr{D}_y')$ des "noyaux semi-compacts en x", de l'espace $\mathscr{E}_x(\mathscr{D}_y') \cap \mathscr{E}_y(\mathscr{D}_x')$ des "noyaux réguliers", etc. Parmi les noyaux réguliers l'A. considère en détail les noyaux de convolution, $S(\xi - \hat{y})$. La classe des noyaux des opérateurs différentiels (i.e. opérateurs de "caractère local") est également étudiée en détail, ainsi que l'espace $L_x^1 \hat{\otimes} L_y^1$ des "noyaux fonctions" (fonctions localement sommables sur $X^l \times Y^m$). Le § 4 finit avec une étude développée de la composition, au sens de Volterra, de noyaux distributions, et avec quelques mots sur les noyaux à valeurs vectorielles.

Enfin, le § 5 est réservé à la théorie de l'intégration de distributions. Les distributions scalaires "sommables" sur R^n sont les éléments de l'espace $\mathscr{D}'_{L^1}$, dual de l'espace $\mathscr{E}$ des fonctions indéfiniment dérivables sur R^n convergeant vers 0 à l'infini ainsi que chacune de leurs dérivées. L'espace des distributions à valeurs dans E, "sommables" sur R^n, sera donc $\mathscr{D}'_{L^1}(E) = \mathscr{D}'_{L^1} \varepsilon E$, et l'intégrale $\int_{R^n} T(x)dx$ d'une distribution T de cet espace est, par définition, $T(1) \in E$, l'espace $\mathscr{D}'_{L^1}$ étant aussi le dual de son dual $\mathscr{B}_c$, constitué par les fonctions $\varphi \in \mathscr{E}$ bornées ainsi que leurs dérivées, et muni de topologie de la convergence compacte dans le dual. Plusieurs propriétés classiques de l'intégrale se généralisent à ce nouveau concept d'intégrale; l'A. reprend l'étude de la convolution, des intégrales de Fourier, etc., à la lumière de cette notion, d'après les lignes générales esquissées dans son Séminaire 1953–54.

J. Sebastião e Silva (Lisbon)

6535:

Tillmann, Heinz Günther. Der Rungesche Approximationssatz und die Theorie der analytischen Funktionale. Math. Ann. **137** (1959), 78–82.

L'A. a montré que, dans la théorie des fonctionnelles analytiques, le "principe des identités" (i.e. "de l'unicité du prolongement analytique") est valable si l'on substitue

à la topologie de Fantappiè une autre, plus fine, métrisable [Abh. Mat. Sem. Univ. Hamburg **21** (1957), 139–193; MR **20** #1911]. Il a généralisé ce résultat au cas des fonctionnelles $F[\varphi]$ à domaine contenu dans l'espace $\mathscr{S}(X)$ des fonctions φ holomorphes dans des ouverts d'une surface de Riemann X (ou de C^n). Maintenant l'A. généralise le même théorème au cas où X est une variété analytique complexe ou "espace complexe", muni d'une "métrique de Runge", c'est-à-dire d'une métrique $d(p, q)$ telle que la "propriété d'approximation de Runge" soit valable sur X, sous la forme généralisée que H. Behnke [*Colloques sur les fonctions de plusieurs variables*, Bruxelles, 1953, pp. 81–96, Thone, Liège, 1953; MR **15**, 696] et K. Stein [ibid., pp. 97–107; MR **15**, 695] ont donné à cette proposition. Réciproquement, l'A. démontre que, si le principe des identités a lieu pour les fonctionnelles analytiques linéaires (il suffit de considérer celles linéaires) dans des régions de $\mathscr{S}(X)$, alors X a une métrique de Runge. Ainsi, le principe des identités pour les fonctionnelles analytiques dans des ouverts $\mathscr{A} \subset \mathscr{S}(X)$ devient équivalent à la propriété de Runge pour les fonctions holomorphes dans des ouverts convenables $D \subset X$. *J. Sebastião e Silva* (Lisbon)

6536:

Thorp, Edward O. The relation between a compact linear operator and its conjugate. Amer. Math. Monthly **66** (1959), 764–769.

If X and Y are two normed linear spaces and T a bounded linear operator with domain X and range $R(T)$ in Y, there are nine possibilities for T, obtained by combining two of the following situations: $R(T) = Y$, $R(T) \neq Y$ but is dense in Y, $R(T)$ is not dense in Y, T^{-1} exists and is bounded, T^{-1} exists but is unbounded, T^{-1} does not exist. Hence there are 81 possibilities, called "states" for the pair (T, T'). Take now a 9 by 9 chessboard and cross out those squares for which the corresponding state is impossible for any pair (T, T') regardless of the choice of X and Y. In this manner we obtain the so-called state diagram for bounded linear operators [cf. A. E. Taylor and C. J. A. Halberg, Jr., J. Reine Angew. Math. **198** (1957), 93–111; MR **20** #1225]. The author constructs an analogous diagram for compact operators (with appreciably fewer white squares) and shows that the diagram for weakly compact operators is identical with that for bounded operators. *V. Pták* (Prague)

6537:

Pachale, Helmut. Über den Urysohnschen Integraloperator. Arch. Math. **10** (1959), 134–136.

Sufficient conditions are stated and proved for a nonlinear operator of the form $K\varphi = \int_E K(s, t, \varphi(s))ds$ to be completely continuous in the space $C^{(0,\lambda)}(E)$ of all Hölder continuous functions (with exponent λ, $0 < \lambda < 1$) defined on a compact set E in Euclidean n-space. The conditions are as follows: (a) $K(s, t, \cdot)$ is continuous for all s in E and almost all t in E; (b) $K(s, \cdot, u)$ is measurable for all s in E and all real u; (c)

$$\int_E \sup_{|u| \leq \gamma} |K(s, t, u)|dt < \infty$$

for all s in E and each $\gamma \geq 0$; (d) there exists a function $k(\eta)$ defined for $0 \leq \eta < \delta$, vanishing at the origin and continuous there, such that

$$\int_E \sup_{|u-v| \leq \eta} |\{K(s_2, t, u) - K(s_2, t, v)\} - \{K(s_1, t, u) - K(s_1, t, v)\}|dt \leq k(\eta)\overrightarrow{s_2 s_1}$$

for all s_1, s_2 in E and arbitrary $\eta \geq 0$; (e) there exists a number $k_0 \geq 0$ such that, for all s_1, s_2 in E,

$$\int_E |K(s_2, t, 0) - K(s_1, t, 0)|dt \leq k_0\overrightarrow{s_2 s_1}.$$

D. H. Hyers (Los Angeles, Calif.)

6538:

Birman, M. Š. Perturbations of quadratic forms and the spectrum of singular boundary value problems. Dokl. Akad. Nauk SSSR **125** (1959), 471–474. (Russian)

Let A and C be two unbounded positive quadratic forms in a Hilbert space H having the same dense domain of definition D, closed in the metrics induced by A and C. Let $\tilde{A}$ and $\tilde{C}$ be the corresponding linear operators. The main general result announced is that $\tilde{A}^{-1} - \tilde{C}^{-1}$ is compact from H to D (in particular, A and C have the same continuous spectrum) provided $A - C$ is majorized by a form B such that $\tilde{A}^{-1}$ is compact from H to D, equipped with the metric induced by B. Applications are given to cases when A and C are integrals

$$\int \sum \varphi_{\alpha\beta}(x) D_\alpha f(x) \overline{D_\beta f(x)} dx$$

over n-space such that the coefficients of $A - C$ tend to zero at infinity. One of them states that the continuous spectrum of the operator $L_h = h(-1)^m (d/dx)^{2m} + p(x)$ $(h, p(x) > 0)$ is the interval $\lambda \geq 0$ for all h if and only if $\int_I p(x)dx \to 0$ as the interval I tends to infinity in such a way that its length is constant. *L. Gårding* (Lund)

6539:

Lidskiĭ, V. B. Conditions for completeness of a system of root subspaces for non-selfadjoint operators with discrete spectrum. Trudy Moskov. Mat. Obšč. **8** (1959), 83–120. (Russian)

Let $\mathfrak{H}$ be a separable Hilbert space, T a linear operator in $\mathfrak{H}$, whose spectrum $\sigma(T)$ is discrete (except in one point at most). Let λ_s $(s = 1, 2, \cdots)$ be the isolated points of $\sigma(T)$, and $\mathfrak{R}_s$ the corresponding spectral invariant subspaces. An important problem related to the spectral theory of such operators is to determine if $\mathfrak{R}_s$ $(s = 1, 2, \cdots)$ generate a dense subspace of $T\mathfrak{H}$. In this direction, the author's principal results are the following two theorems. (1) Let T be a Hilbert-Schmidt operator (i.e. T is bounded, defined on $\mathfrak{H}$ and $\sum_{i=1}^{\infty} (T\phi_i, T\phi_i) < +\infty$, where $\{\phi_i\}$ is an orthonormal basis in $\mathfrak{H}$), so that $\frac{1}{2}(T + T^*)$ and $-\frac{1}{2}i(T - T^*)$ are ≥ 0 or ≤ 0. Then the subspaces $\mathfrak{R}_s$ $(\lambda_s \neq 0)$ generate a dense subspace of $T\mathfrak{H}$. (2) Let L_1 and L_2 be symmetric operators defined on a dense subspace $\mathfrak{D}$ of $\mathfrak{H}$ so that $(L_1 + iL_2 - \lambda_0 E)\mathfrak{D}$ is dense in $\mathfrak{H}$ for a λ_0. Suppose also that one of the following conditions is verified. (a) L_1 and L_2 are semi-bounded and, for one of the three Friedrichs self-adjoint extensions $\tilde{L}_1$, $\tilde{L}_2$ or $(L_1 + L_2)^{\sim}$, the resolvent is a Hilbert-Schmidt operator. (b) The resolvent of $\tilde{L}_1$ is a Hilbert-Schmidt operator and

$$(L_1 f, f) - |(L_2 f, f)| \geq -\sigma^2(f, f)$$

for all $f \in \mathfrak{D}$ (where σ is a fixed real number $\neq 0$). Then the operator $L = L_1 + iL_2$ can be extended in a closed linear operator $\bar{L}$, whose spectrum $s(\bar{L}) = \{\lambda_s\}$ is discrete and $\mathfrak{R}_s$ $(s = 1, \cdots)$ generate a dense subspace of $\mathfrak{H}$.

As an application is given a condition on strongly elliptic operators L (in Višik's sense [Dokl. Akad. Nauk SSSR **74** (1950), 881–884; MR **13**, 134]) of order $2m$ in a domain D of R^n, with $4m > n$, so that the boundary problem

$$Lu = \lambda u,$$

$$u\Big|_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma = \cdots = \frac{\partial u^{m-1}}{\partial n^{m-1}}\Big|_\Gamma = 0,$$

where $\Gamma = \operatorname{Fr} D$, considered in $L^2(D)$, determines an operator $\bar{L}$ having the properties of the above theorem.

C. Foiaş (Bucharest)

6540:

Browder, Felix E. Functional analysis and partial differential equations. I. Math. Ann. **138** (1959), 55–79.

Partial differential equations are present in this paper (part I) only as hidden motivators of the discussion. The formal content is a series of propositions on linear operators, phrased in the very general setting of partially defined discontinuous linear operators on locally convex spaces.

Existence theorems for partial differential equations frequently take the form that the range of a certain operator is closed, a property that is given detailed study in sections 1 and 2. Let T be a linear operator, densely defined on the locally convex space E with range in the locally convex space F. Theorem 1.1 says in part that if E and F are Banach spaces and T is closed then the range of T is closed in F if and only if the range of T^* is closed in E^*. (This was proved independently by G.-C. Rota in his 1956 Yale thesis.) This is established as a special case of the following more general assertion (Theorem 2.1, V, in part): If F^*, $(E \times F)^*$, and the graph of T are fully complete and if F, the graph of T^*, and the dual of the graph of T are fully tonnelé then the range of T is closed if and only if the range of T^* is closed. Many related results are established.

In section 3 the author presents an abstract theory of boundary value problems. Let T_0 and T_0' be formally adjoint; i.e., $\langle T_0 e, f^* \rangle = \langle e, T_0' f^* \rangle$ for all e in $D(T_0) E \subseteq$ and f^* in $D(T_0') \subseteq F^*$. The operator T_1 is defined as the restriction of the adjoint of T_0' to those elements in its domain which lie in E and are sent into F. One wishes to find a [completely] solvable operator T for (T_0, T_0'); i.e., a closed densely defined operator $T_0 \subseteqq T \subseteqq T_1$ whose inverse is a [completely] continuous operator defined on all of F. Necessary and sufficient conditions for the existence of a solvable operator for (T_0, T_0') are given for the case of Fréchet spaces, reflexive Banach spaces, and Hilbert spaces.

A thread which runs through the paper is a discussion of operators which are semibounded in some general sense. The author shows that such an operator T, if closed and densely defined on a Banach space E, has a closed range and bounded inverse, and the range of T^* is E^* (Theorem 1.5), and proves related results. Finally, if T_0 is densely defined on the locally convex space E with range in F, and if J is a linear mapping of $D(T_0)$ onto a dense subset of F^* such that $\operatorname{Re} \langle T_0 e, Je \rangle > 0$ for $e \neq 0$ in $D(T_0)$, he defines an inner product

$$[e, e_1] = \frac{1}{2}\{\langle T_0 e, Je_1 \rangle + \overline{\langle T_0 e_1, Je \rangle}\}$$

and shows under suitable assumptions that T_0 has a generalized Friedrichs extension (Theorem 3.6).

E. Nelson (Princeton, N.J.)

6541:

Kato, Tosio. Perturbation theory for nullity, deficiency and other quantities of linear operators. J. Analyse Math. **6** (1958), 261–322.

This incisive paper studies, among many other things, the effect of perturbation on the dimensions of the null space and the deficiency space of unbounded closed linear operators between Banach spaces. It has many points of contact with I. C. Gohberg and M. G. Krein [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2 (74), 43–118; MR **20** #3459] but the methods are quite different. The main ideas, at least for bounded operators, go back to J. A. Dieudonné [Bull. Sci. Math. (2) **67** (1943), 72–84; MR **7**, 124], F. V. Atkinson [Mat. Sb. (N.S.) **28** (**70**) (1951), 3–14; MR **13**, 46] and B. Yood [Duke Math. J. **18** (1951), 599–612; MR **13**, 355] (cf. GK = Gohberg-Krein for other references). However, the present author has introduced a number of new quantities and ideas.

Let X, X' be complex Banach spaces. If M is a linear manifold in X, then dim M is the algebraic dimension of M and takes one of the values $0, 1, 2, \cdots, \infty$. (The paper of GK defines dimension topologically and considers arbitrary cardinals.) The first part of this paper presents some new results concerning linear manifolds and "orthogonal elements" in X and is similar to the discussion in GK concerning the opening between two manifolds. In the following, let A be a closed linear operator with domain $D(A)$ (not necessarily dense) in X, range $R(A)$ in X' and null space $N(A)$. The nullity $\alpha(A)$ of A is defined to be dim $N(A)$ and the deficiency $\beta(A)$ of A is dim $X'/R(A)$. (This differs somewhat from the definitions in GK.) One of the main problems is to consider the variation of $\alpha(A)$ and $\beta(A)$ under the perturbation of A by somewhat "smaller" operators. The author introduces a constant $\gamma(A)$, called the lower bound of A, which is defined to be the supremum of all numbers $\gamma \geqq 0$ such that $\|Ax\| \geqq \gamma \|\tilde{x}\|$, $x \in D(A)$, where $\tilde{x}$ is the coset $x + N(A)$ and $\|\tilde{x}\|$ denotes the usual factor space norm in $X/N(A)$. Others have considered this constant before [cf. the reviewer's note, Ann. Acad. Sci. Fenn. Ser A. I. no. 257 (1958); MR **21** #2932], but this reviewer is not aware of any previous systematic use of $\gamma(A)$. The range $R(A)$ is closed if and only if $\infty \geqq \gamma(A) > 0$. For simplicity, assume here that $D(A)$ is dense in X so that the adjoint operator A^* is defined. It is proved that $\gamma(A) = \gamma(A^*)$. This has as a consequence the important result that $R(A)$ is closed if and only if $R(A^*)$ is closed. This is classical for bounded A and was stated and partially proved by G.-C. Rota [Comm. Pure Appl. Math. **11** (1958), 23–65; MR **20** #3334], whose proof was completed by J. T. Joichi [Univ. of Illinois dissertation, 1959]. It also follows from a theorem of S. Goldberg [#6542 below] and the question was examined in great generality and detail by F. E. Browder [#6540]. The author's proof is very direct. Next closed linear operators with closed ranges and their products are considered; here some contact and overlap is made with the work of Joichi [ibid.].

It is proved that if A is a closed linear operator with closed range and at least one of $\alpha(A)$, $\beta(A)$ is finite, and if B is a bounded operator with $D(B) \supset D(A)$ and $\|B\| < \gamma(A)$, then the closed operator $A+B$ has closed range, and $\alpha(A+B) \leq \alpha(A)$, $\beta(A+B) \leq \beta(A)$ and $\alpha(A+B) - \beta(A+B) = \alpha(A) - \beta(A)$. Except for the condition $\|B\| < \gamma(A)$, which is of some interest: this result is known from GK (§ 7). It is shown that the theorem remains true for certain unbounded operators B such that

$$(*) \quad D(B) \supset D(A), \quad \|Bx\| \leq \sigma\|x\| + \tau\|Ax\| \quad (x \in D(A)),$$

provided that σ, τ are non-negative and satisfy $\sigma + \tau\gamma(A) \leq \gamma(A)$. Also the result fails, in general, if both $\alpha(A)$, $\beta(A)$ are infinite.

A closed and bounded operator B is said to be strictly singular in case, given a closed linear manifold M in $D(B)$, the existence of a constant $c > 0$ such that $\|Bx\| \geq c\|x\|$, $x \in M$, implies that $\dim M < \infty$. The set of everywhere defined strictly singular operators is proved to form a two-sided ideal which is closed in the uniform operator topology and which contains the ideal of completely continuous operators. (These two ideals coincide in the case of Hilbert space.) It is proved that if A has closed range and $\alpha(A) < \infty$ and if B is strictly singular, then $A+B$ has closed range and $\alpha(A+B) < \infty$. This was proved in GK for completely continuous B. Extensions are given to certain unbounded operators B.

Define the manifolds $M_n = (B^{-1}A)^n(X)$, $N_n = (A^{-1}B)^n(0)$, $n = 0, 1, 2, \cdots$; here B^{-1} and A^{-1} may be many-valued. By $\nu(A:B)$ is meant the smallest integer (or ∞) such that $N_1 = N(A)$ is not a subset of M_n (or, equivalently, for which N_n is not a subset of M_1). In the case $\nu(A:B) = \infty$, perturbation of the closed non-zero operator A with closed range by certain unbounded operators B is considered. This gives information in the case where α, β may both be infinite on A. It is also seen that under certain conditions $\gamma(A - \lambda B)$ satisfies a Lipschitz condition in λ.

Let A be closed with closed range and let B be bounded and closed. If $\nu(A:B)$ is finite and one of $\alpha(A)$, $\beta(A)$ is finite, then the Banach spaces X, X' admit decomposition into the direct sums of a finite number of finite-dimensional subspaces and a "residual" part on which the corresponding $\nu = \infty$. The finite-dimensional parts behave like the manifolds in the Jordan reduction of a matrix. Under the same hypotheses on A and hypothesis (*) on B, it is proved that if $0 < |\lambda| < \rho$, for some ρ, then $A - \lambda B$ is such that $\nu(A - \lambda B : B) = \infty$. Hence the finiteness of ν is not a stable property.

Next the author dissects the complex plane into several parts giving the spectral character of $A - \lambda B$ in somewhat more detail than is usual when $X = X'$ and $B = I$. As a consequence of this he obtains the theorem that if B is strictly singular and $\lambda \neq 0$, then $B - \lambda I$ has closed range and $\alpha(B - \lambda I) = \beta(B - \lambda I) = 0$ except for a countable number of discrete eigenvalues where $0 < \alpha(B - \lambda I) = \beta(B - \lambda I) < \infty$. This generalizes a classical theorem of F. Riesz and J. Schauder for completely continuous operators.

R. G. Bartle (Urbana, Ill.)

6542:

Goldberg, Seymour. Ranges and inverses of perturbed linear operators. Pacific J. Math. **9** (1959), 701–706.

Properties like density or non-density of range of a linear operator T are shown to be invariant under addition of a linear operator of small norm, even in non-complete normed linear spaces. *J. T. Schwartz* (Berkeley, Calif.)

6543:

Iohvidov, I. S.; and Kreĭn, M. G. Spectral theory of operators in spaces with indefinite metric. II. Trudy Moskov. Mat. Obšč. **8** (1959), 413–496. (Russian)

Continuing their previous work [see same Trudy **5** (1956), 367–432; MR **18**, 320; for notations, etc.] the authors begin with a discussion of invariant subspaces of unitary operators in $\mathfrak{P}_n$. (In particular an amendment is offered to an incorrectly stated and incorrectly "proven" theorem.) For example: (i) All maximal positive invariant subspaces of a unitary operator U have the same (finite) dimension, and the eigenvalues and corresponding multiplicities of U restricted to each such subspace are the same; (ii) to each eigenvalue of a cyclic unitary operator there is exactly one (up to scalar multiples) eigenvector and, if all eigenvalues of U are of absolute value 1, there is a unique n-dimensional eigenspace (n is the dimension of the maximal positive subspace of Π_n); (iii) if P is a polynomial with no roots inside the unit circle, then $(P(U)x, P(U)x) \leq 0$ for all x in Π_n if and only if P is divisible by the minimal polynomial of U in $\mathfrak{T}$.

In the following section there is an extension of Herglotz's theorem on positive definite sequences to indefinite sequences. A sequence $\{c_p\}_0^\infty$, $c_p = \bar{c}_p$ is said to be in $\mathfrak{P}_n$ if there is an n_0 such that every form $\sum_{p,q=0}^{N-1} c_{p-q}\xi_p\bar{\xi}_q$ has exactly n positive square terms for $N \geq n_0$. Theorem: $\{c_p\}_0^\infty$ is in $\mathfrak{P}_n$ if and only if

$$c_p = \sum_{j=1}^{r} (Q_j(ip)\lambda_j^p + \bar{Q}_j(ip)\bar{\lambda}_j^{-p}) + \sum_{k=1}^{s} R_k(ip)e^{ip\varphi_k} - \int_{-\pi}^{\pi} \frac{e^{pit} - S_p(t)}{\prod_{k=1}^{s}\left(\sin\frac{t-\varphi_k}{2}\right)^{2\rho_k}}\, d\sigma(t),$$

where: (a) $\sigma(t) = \sigma(t-0)$, $\sigma(-\pi) = 0$, $\sigma \uparrow$ on $(-\pi, \pi)$; (b) $\rho_k \geq 0$, $-\pi \leq \varphi_k < \pi$; (c) $|\lambda_j| > 1$; (d) R_k is a polynomial of degree r_k, $0 \leq r_k < 2\rho_k - 1$; (e) Q_j is a polynomial of degree $q_j - 1$, where $\sum_j q_j + \sum_k \rho_k = n$; (f) S_p is a convergence producing term for the integral, e.g., so that $e^{pit} - S_p(t)$ has a zero of order at least $2\rho_k$ for $t = \varphi_k$. For example, there is a unique polynomial that accomplishes this.

The proof depends upon these constructions: (i) Π: all sequences $\{\xi_p\}_{-\infty}^{\infty}$ almost all terms of which are 0, with scalar product $\sum_{p,q=-\infty}^{\infty} c_{p-q}\xi_p\bar{\eta}_q$; (ii) U, a cyclic unitary operator in Π: $U\{\xi_p\} = \eta_p$, where $\eta_p = \xi_{p-1}$, and its minimal polynomial

$$P_n(\lambda) = \prod_{j=1}^{r} (\lambda - \lambda_j)^{p_j} \prod_{k=1}^{s} (\lambda - \varepsilon_k)^{n_k},$$

where $\sum_j p_j + \sum_k \rho_k = n$; (iii) L, a linear mapping: $L(c_p) = \sum_{j=-n}^{n} \alpha_j c_{p+j}$ where $\sum_{j=-n}^{n} \alpha_j\lambda^j = -P_n(\lambda)P_n^*(\lambda)$, ($P_n^*(\lambda)$ arises from $P_n(\lambda)$ by replacing λ_j with $\bar{\lambda}_j^{-1}$); $\{L(c_p)\} = \{d_p\}$ is a positive definite sequence with Herglotz function $\omega(t)$;

$$\sigma(t) = \text{const} \int_{\pi}^{t} \prod_{j=1}^{r} \left(\sin^2\frac{\tau - \varphi_j}{2} + \sinh^2\frac{\xi_j}{2}\right)^{p_j} d\omega(\tau).$$

Subject to certain normalizations, the above solution is unique. If a representation is given, the number n can be determined, and criteria for the boundedness of $\{c_p\}$ are

given. The rank of the sequence $\{c_p\}$ in terms of the ranks of its associated forms is discussed. Finally, the problem of extending a finite sequence $\{c_p\}_0^{N-1}$ to an infinite sequence in $\mathfrak{P}_n$ is solved and the extension sequence is shown to be unique. Special cases are considered.

The last section treats helices in infinite-dimensional Lobachevsky spaces. In any metric space a curve x_t is called a helix if $\rho(x_s, x_t)=\Delta(t-s)$ where ρ is the metric, and Δ is some function. In Π_1 let e be a vector such that $(e, e)=1$ and let $L=\{x|(x, x)=1, (x, e)>0\}$. Then L is a connected set in Π_1, and for arbitrary R, L can be metrized by the distance function $d(x, y)=R\ln[(x, y)+((x, y)^2-1)^{1/2}]$. So metrized, L is a complete metric space L_R. Theorem: $\Delta(t)$ $(-T<t<T)$ is the metric function of a helix in L_R if and only if $f(t)=\cosh(\Delta(t)/R)$ satisfies one of the following conditions.

$$\text{(I)} \qquad f(t) = \rho\cosh\zeta t - \int_0^\infty \cos t\lambda\, d\sigma_\Gamma(\lambda),$$

where $\rho, \zeta>0$, $\sigma_\Gamma(\lambda)=\sigma_\Gamma(\lambda-0)$, $\sigma_\Gamma\uparrow(0\leqq\lambda<\infty)$, $\sigma_\Gamma(0)=0$ and $\rho=\int_0^\infty d\sigma_\Gamma(\lambda)+1$.

$$\text{(II)} \qquad f(t) = 1+\int_0^\infty \frac{1-\cos\lambda t}{\lambda^2}\, d\sigma_\Pi(\lambda),$$

where $\sigma_\Pi(\lambda)=\sigma_\Pi(\lambda-0)$, $\sigma_\Pi\uparrow$ $(0\leqq\lambda<+\infty)$, $\sigma_\Pi(0)=0$ and $\int_1^\infty(1/\lambda^2)d\sigma_\Pi(\lambda)<\infty$. The connection with the Bochner version of the Herglotz theorem is similar to that between the authors' results on sequences and the Herglotz theorem. (Cases I and II with σ_Γ and σ_Π are the "hyperbolic" and "parabolic" cases, respectively.) The problem of extension of the metric function Δ of a helix defined on a finite open interval to an infinite interval is solved.

B. Gelbaum (Minneapolis, Minn.)

6544:

Kendall, David G. Sur quelques critères classiques de compacité dans certains espaces fonctionnels, et la théorie des semi-groupes de transformations. J. Math. Pures Appl. (9) **38** (1959), 235–244.

Let $G=[T_t; t\geq 0]$ denote a strongly continuous and bounded semi-group of linear bounded operators on a Banach space X; let Ω denote the infinitesimal generator of G, J_λ the resolvent of Ω at λ, and let G^* denote the adjoint semi-group. A semi-group G is said to provide a "criterion of compactness" if, for each bounded set A, $|T_tx-x|\to 0$ as $t\to 0$ uniformly on A if and only if the set A is conditionally compact. The author proves the following result. (i) The semi-group G provides a criterion of compactness if and only if the resolvent J_λ is compact for one (and hence for all) values of $\lambda>0$. (ii) The semi-group G provides a criterion of compactness if and only if the same is true of the adjoint semi-group G^*. Applying this to the group of translates of the continuous functions on the circumference C of the unit circle, $(T_tf)(x)=f(x+t)$, one sees from the d'Arzela-Ascoli theorem that G provides a criterion of compactness. As a consequence the same is true of the adjoint semi-group and this is shown to be a restatement of the M. Riesz compactness criterion in $L(C)=X^*$. For a semi-group G in a space of type (L) or (M) it is proved that the following conditions are equivalent: (a) G is reflexive in the sense of adjoint semi-groups; (b) G provides a criterion of compactness; and (c) J_λ is compact.

R. S. Phillips (Los Angeles, Calif.)

6545:

Newman, Donald J. A radical algebra without derivations. Proc. Amer. Math. Soc. **10** (1959), 584–586.

Singer and Wermer [Math. Ann. **129** (1955), 260–264; MR **16**, 1125] proved that a bounded (=continuous) derivation in a commutative Banach algebra maps the algebra into its own radical. In the present paper a Banach algebra which is all radical and which has no non-trivial bounded derivations is exhibited.

D. A. Edwards (Newcastle-upon-Tyne)

6546:

Sankaran, S. The *-algebra of unbounded operators. J. London Math. Soc. **34** (1959), 337–344.

A definition of local measurability of an operator with respect to a ring of operators is given in this paper, generalizing the notion of measurability given by Segal [Ann. of Math. (2) **57** (1953), 401–457; **58** (1953), 595–596; MR **14**, 991; **15**, 204]. Theorem: The collection of all operators locally measurable with respect to a ring of operators having a countably decomposable center is a self-adjoint algebra with respect to strong sum and strong product. This extends a result of Murray and von Neumann [Ann. of Math. (2) **37** (1936), 116–229], and of Segal, op. cit. For a purely infinite ring, an operator is measurable if and only if it is in the ring. This is not true for locally measurable operators, as an example shows.

F. B. Wright (New Orleans, La.)

6547:

Tomiyama, Jun. On the projection of norm one in W^*-algebras. III. Tôhoku Math. J. (2) **11** (1959), 125–129.

[For parts I and II, see Proc. Japan Acad. **33** (1957), 608–612; Tôhoku Math. J. (2) **10** (1958), 204–209; MR **20** #2635; **21** #2928.] Let M and N be W^*-algebras. Let M_* be the space of all σ-weakly continuous linear functionals on M. A positive linear functional ϕ on a W^*-algebra is called singular if for no non-zero positive linear functional γ is $\gamma\leqq\phi$. M_*^1 is the closed subspace of M^* generated by the singular linear functionals. A continuous linear mapping $\pi: M\to N$ is singular if ${}^t\pi(N_*)\subset M_*^1$, where ${}^t\pi$ is the transpose of π. $\pi=\pi_1+\pi_2$, where π_1 is σ-weakly continuous, π_2 is singular, and the representation is unique. If $\pi: M\to N$ is a positive linear mapping, then π is singular if and only if there is no non-zero positive, linear, normal map π' such that $\pi'\leqq\pi$. If there is a projection π of norm one from a semi-finite W^*-algebra M to its purely infinite W^*-subalgebra N, then π is singular. If there is a projection π of norm one from a W^*-algebra M of type I to its W^*-subalgebra N of type II, then π is singular. If M is a W^*-factor of type I and N is its W^*-subalgebra, then a necessary and sufficient condition that there be a σ-weakly continuous projection of norm one from M to N is that N be the second dual of a two-sided ideal.

S. Sherman (Philadelphia, Pa.)

6548:

Berberian, S. K. Note on a theorem of Fuglede and Putnam. Proc. Amer. Math. Soc. **10** (1959), 175–182.

For linear operators on a Hilbert space, Fuglede [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 35–40; MR **11**, 371] proved that: a normal, $ba=ab$ implies $ba^*=a^*b$; and Putnam [Amer. J. Math. **73** (1951), 357–362; MR **12**, 717] obtained the generalization: $ba_1=a_2b$ implies $ba_1^*=a_2^*b$

if a_1 and a_2 are normal. The author calls a ring A with involution an FT-ring [resp. PT-ring] when Fuglede's [resp. Putnam's] theorem holds in A. He proves that A is a PT-ring if A_2 is an FT-ring, where A_2 is the ring of all 2×2 matrices over A. Putnam's unitary equivalence theorem (similar normal elements are unitarily equivalent) is also generalized for a PT-ring A if A moreover satisfies "the square root axiom": For any $a\in A$ there is $r=r^*\in A$ such that $r^2=a^*a$ and r is in the double commutant of a^*a. In the remaining half, the author is mainly concerned with a finite AW^*-algebra A and its regular ring C, which he introduced earlier [Ann. of Math. (2) **65** (1957), 224–240; MR **18**, 914], and proves that the following properties hold in C if they hold in A: (1) The subnormality of Halmos [Summa Brasil. Math. **2** (1950), 125–134; MR **13**, 359] coincides with normality; (2) every increasing net of self-adjoint elements bounded above has a least upper bound. (2) is an AW^*-extension of a theorem of Ogasawara and Yoshinaga [J. Hiroshima Univ. Ser. A **19** (1955), 273–299; MR **18**, 119]. *M. Nakamura* (Asiya)

6549:

Goldman, Malcolm. On subfactors of factors of type II_1. Michigan Math. J. **6** (1959), 167–172.

Let f be a cyclic trace vector of a II_1-factor $\mathscr{M}$ and its commutant $\mathscr{M}'$. If $\mathscr{A}$ is a subfactor of $\mathscr{M}$ with the finite commutant $\mathscr{A}'$, then $c=\dim_{\mathscr{A}'}[\mathscr{A}f]$ exists with $0<c<1$. The author proves that there exists a partial isometry V in $\mathscr{M}$ such that $\dim(V^*V)=c$ and $g=aVf$ $(a>0)$, where g is a trace vector for $\mathscr{A}$. Using this, the author proves finally, for $c=\frac{1}{2}$, that each $M\in\mathscr{M}$ is expressible as $A+BU$, where A and B belong to $\mathscr{A}$ and U is a unitary operator of $\mathscr{M}$ with $U^2=I$. The paper also contains a counterexample due to McLaughlin which violates the decomposition of $\mathscr{M}$ for $c=\frac{1}{3}$. It seems to the reviewer that, for $c=\frac{1}{2}$, $\mathscr{M}$ is the crossed product $G\otimes\mathscr{A}$ recently studied by the reviewer and Z. Takeda [review below], where $G=\{I, U\}$ and the action of U on $\mathscr{A}$ is defined by $A\to UAU$. It is not hard to see that the action of U defines an outer automorphism of $\mathscr{A}$.

M. Nakamura (Asiya)

6550:

Nakamura, Masahiro; and Takeda, Zirô. On some elementary properties of the crossed products of von Neumann algebras. Proc. Japan Acad. **34** (1958), 489–494.

The authors discuss von Neumann algebras in the context of crossed products of groups and rings. Their examples yield factors of crossed products of enumerable groups of outer automorphisms of subfactors.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

6551:

Nakamura, Masahiro; and Takeda, Zirô. On certain examples of the crossed product of finite factors. I. Proc. Japan Acad. **34** (1958), 495–499.

The authors discuss von Neumann factors of finite type as crossed products of torsion-free automorphism groups of factors with factors. It is shown that every enumerable infinite group is an automorphism group of a hyperfinite continuous factor. *E. L. Griffin, Jr.* (Ann Arbor, Mich.)

6552:

Nakamura, Masahiro; and Takeda, Zirô. On certain examples of the crossed product of finite factors. II. Proc. Japan Acad. **34** (1958), 500–502.

This continuation of the authors' first paper shows that every infinite group is an outer automorphism group of a hyperfinite continuous factor.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

6553:

Nicolescu, Lilly Jeanne. Propriétés des fonctions réelles directement différentiables de Gâteaux, du second ordre. Com. Acad. R. P. Romîne **9** (1959), 415–418. (Romanian. Russian and French summaries)

The author extends three theorems (Rolle, Cauchy, Valnberg) to real-valued functions f defined on a real normed vector space which are twice Gâteaux differentiable. *E. R. Lorch* (New York, N.Y.)

CALCULUS OF VARIATIONS

6554:

Uspenskiĭ, S. V. A variational problem of Hilbert. Dokl. Akad. Nauk SSSR **127** (1959), 526–528. (Russian)

The author states results concerning existence, uniqueness, and smoothness of the solutions of a problem of minimum presenting a fixed closed smooth surface S of discontinuity. The minimum of an integral $G[g]$ (given below) is sought in the class M of all functions $g(P)$, $P\in R$ (R the n-dim. Euclidean space, $n>2$, r the distance of P from the origin), with generalized first partial derivatives g_i, $i=1,\cdots,n$, with generalized limit zero as $r\to\infty$, satisfying $F[g]<\infty$ (F given below) and the boundary condition (*) $ag_+-bg_-=h$ on S, where g_+, g_- denote the limit values of g at Q, taken along the normal to S at Q, $Q\in S$, respectively inside and outside S, and $a(Q)$, $b(Q)$, $h(Q)$, $Q\in S$, are given functions defined on S and sufficiently smooth (h Lipschitzian on S of exponent $\frac{1}{2}+\varepsilon$, $\varepsilon>0$). The functionals F, G are of the form

$$F[g]=(R)\int\Big(\sum_{ij}A_{ij}(P)g_ig_j+B(P)g^2\Big)dR,$$

$$G[g]=F[g]-2(R)\int fg\,dR,$$

where A_{ij}, B are given smooth functions in R with $\sum A_{ij}(P)u_iv_j>c\sum_i u_i^2$ for some constant $c>0$, all real $u_1,\cdots,u_n$ and $P\in R$, and $f(P)$, $P\in R$, is a given function required to satisfy $f(1+r)^{1+\varepsilon}$ of class L^2 in R. The minimum of $G[g]$ in the class M is attained by one (and only one) function U, and U has generalized second partial derivatives, satisfies the corresponding second order Euler equation and the expected boundary condition. The results are only stated. Analogous results are stated when the boundary condition (*) is replaced by a nonlinear boundary condition of the form $g_+=F(g_-,Q)+h(Q)$, $Q\in S$. The results improve previous ones by S. M. Nikol'skiĭ [Mat. Sb. (N.S.) **33** (**75**) (1953), 261–326; Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 599–630; MR **16**, 453; **20** #7235]. Pertinent reference: S. L. Sobolev, *Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike*, Izdat. Leningrad. Gos. Univ., 1950 [MR **14**, 565].

L. Cesari (Madison, Wis.)

GEOMETRY

See also 6565, 6569, 7060.

6555:

MacLane, Saunders. Metric postulates for plane geometry. Amer. Math. Monthly **66** (1959), 543–555.

The aim of this paper is to exhibit a system of axioms for Euclidean plane geometry from which that geometry can be developed relatively quickly. This end is attained by referring to properties of real numbers in the axioms.

The primitive notions are point, distance, line and angle measure. There are four axioms for distance, those for characterizing a non-trivial semi-metric space. Betweenness, intervals and rays are defined in terms of distance. Next come four line axioms, which lead to coordinate systems for lines and one-dimensional geometry. The next four axioms attribute angle measure to directed angles. They also, implicitly, distinguish an orientation of the plane. Similarity of triangles is defined, as a relation involving the lengths of their sides and the measures of their angles. The side-angle-side test for the similarity of triangles is introduced as the next axiom. The last axiom is a continuity axiom, which connects the distance function with the angle measure function. A sketch is then given of the derivation of incidence and separation properties of the plane, and of a treatment of parallels and perpendiculars. *H. Levi* (New York, N.Y.)

6556:

★Bachmann, Friedrich. Axiomatischer Aufbau der ebenen absoluten Geometrie. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 114–126. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

The construction sketched here is detailed in the book reviewed below.

6557:

★Bachmann, Friedrich. Aufbau der Geometrie aus dem Spiegelungsbegriff. Die Grundlehren der mathematischen Wissenschaften, Bd. XCVI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. xiii+311 pp. DM 49.80.

The construction of plane geometry based on the concept of "reflection" which was originated by J. Hjelmslev and developed by K. Reidemeister, Arnold Schmidt, the author and others, is here presented for the first time in the form of a textbook. The book is written in the tradition of earlier geometry texts in the Grundlehren series; there is a very leisurely introduction; and even if the pace increases after this, it remains pedestrian (without ever becoming pedantic); there is no stingy saving of space by using forced shortcuts; on the contrary, detours are used whenever they shed light on the main topic; and the geometer will enjoy the fact that, in spite of the use of the axiomatic method and the reduction to group-theoretical or analytical forms, the truly geometrical aspect is never lost sight of—witness the 160 figures in the text. The value of the book is increased by numerous exercises, and by careful references at the end of each section. In chapter I the concept of "metric plane" is introduced which is to serve as a common basis of euclidean, elliptic and hyperbolic planes. There are two primitive concepts: "point" and "line"; and two primitive relations: "incidence" $A \mathrel{I} a$ of a point A and a line a, and "orthogonality" $a \perp b$ of two lines a, b. The axioms are: (1) Incidence: There exists a line. Every line contains at least three points. Through two distinct points there is at least one line. (2) Orthogonality: If $a \perp b$ then $b \perp a$. Orthogonal lines have a common point. Through every point A there is to every line a at least one perpendicular; and if $A \mathrel{I} a$ only one. (3) Reflections: There exists a reflection in each line (a reflection in a line a is an involutory one-to-one map of the set of all points onto itself and the set of all lines onto itself which preserves incidence and orthogonality and leaves all points of a fixed; the definition of reflection in a point is dual). The resultant of three reflections in lines a, b, c with either a common point or a common perpendicular is a reflection in a line d (axiom of the three reflections). There are no relations or axioms of order; free mobility is not assumed. The reflections generate the group G of motions, and there is a one-to-one correspondence between the lines and the reflections in lines, and between the points and the reflections in points. This invites a formulation of the axioms entirely within the group of motions, and leads to the following definition of "metric (absolute) geometry" (chapter II). Let G be a group; write $\sigma|\rho$ if σ, ρ and $\sigma\rho$ are involutory elements of G. Let G be generated by an invariant set $S=\{a, b, \cdots\}$ of involutory elements. Denote by $P, Q, \cdots$ those elements of G which can be written in the form ab where $a, b \in S$ and $a|b$. G defines a metric geometry (in which $a, b, \cdots$ correspond to lines, $P, Q, \cdots$ to points, $P|g$ to $P \mathrel{I} g$ and $g|h$ to $g \perp h$) if the following axioms hold: (1) To any P, Q there exists g such that $P|g, Q|g$. (2) If $P|g$, $P|h$, $Q|g$, $Q|h$ then $P=Q$ or $g=h$. (3) If $a|P$, $b|P$, $c|P$ then there exists d such that $abc=d$. (4) If $a|g$, $b|g$, $c|g$ then there exists d such that $abc=d$. (5) There exist g, h, j such that neither $g|h$ nor $j|g$ nor $j|h$ nor $j|gh$. G satisfies these axioms if and only if it is the group of motions of a metric plane. The connection with the classical theory is given by the Main Theorem: Every metric plane can be embedded in a projective-metric plane; and the group of motions of the metric plane is a subgroup of the group of motions of the projective-metric plane (an ordinary projective-metric plane is a projective plane with a given projective polarity, orthogonality of lines being conjugacy; a singular projective-metric plane is a projective plane with a given line g_∞ and an elliptic involution on g_∞; here two lines are orthogonal if they meet g_∞ in corresponding points of the involution). The proof of the Main Theorem is perhaps the most subtle part of the book; the main tool is certain maps called "semi-rotations" (which are not motions) introduced first by Hjelmslev in the non-elliptic case.—In chapter III, the groups of motions of the projective-metric planes are studied as orthogonal groups in metric vector spaces, emphasis being placed on their generation by involutory elements. Chapters IV, V and VI give independent developments of the euclidean, hyperbolic and elliptic geometries defined as metric geometries satisfying appropriate additional axioms. These planes do not exhaust all metric planes; but the problem of determining all metric

planes contained in a given projective-metric plane is as yet unsolved. Some special results and problems in this direction are discussed in an appendix.

F. A. Behrend (Melbourne)

6558:

★Anspach, Pierre A. L. Aperçu de la théorie des polygones réguliers. IV. Published by the author, Brussels, 1959. pp. 299–430.

The study of the regular heptagon and the regular enneagon is continued by the author in this, the fourth, part of his work [1955, 1956; MR **17**, 397, 1122; **18**, 501]. Here he associates with those polygons ten conics, the lemniscate of Bernoulli, conchoids, and cycloids. The book contains eighteen elaborate, full-page figures, but there is no index and no bibliographical references.

N. A. Court (Norman, Okla.)

6559:

Weston, J. D. Volume in vector spaces. Amer. Math. Monthly **66** (1959), 575–577.

An axiomatic definition of the volume of a parallelepiped in the n-dimensional real vector space is proposed which is similar to that given in the book by O. Schreier and E. Sperner, *Einführung in die analytische Geometrie und Algebra*, Bd. I [Teubner, Leipzig-Berlin, 1931, § 8].

H. Schwerdtfeger (Montreal, P.Q.)

6560:

Ferla, Ambrogio. Sulle isomerie vettoriali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 437–460.

Author's resumé: "Vengono stabilite varie proprietà, alcune già note ed altre nuove, delle isomerie vettoriali (che equivalgono alle sostituzioni ortogonali); esse sono state ottenute in modo semplice e rapido coi metodi del calcolo vettoriale omografico." The actual reading of this purely expository paper is difficult and in details impossible for any one who is not used to the notations of vector algebra as introduced in the books *Geometria differenziale* by Burgatti, Boggio and Burali-Forti [Zanichelli, Bologna, 1930] and *Analisi vettoriale generale*, Vol. I, by Burali-Forti and Marcolongo [Zanichelli, Bologna, 1929]. Matrix notations are carefully avoided. Reflections are introduced in an artificial manner, but the theorem concerning the representation of a rotation as a product of reflexions is proved only for $n=2$ and $n=3$ with a considerable amount of symbolism.

H. Schwerdtfeger (Montreal, Que.)

6561:

Schuster, Seymour. Pencils of null polarities. Canad. J. Math. **11** (1959), 614–620.

This paper defines and discusses, using synthetic methods, the pencil of null-polarities in projective 3-space which may be defined analytically by the two matrices $[p_{ij}]$, $[q_{ij}]$, $i, j = 1, 2, 3, 4$, in which

$$p_{12} = p_{23} = p_{34} = q_{12} = q_{23} = q_{34} = q_{14} = 0$$

and the other elements are arbitrary.

T. G. Room (Sydney)

6562:

Sce, Michele. Sulla completezza degli archi nei piani proiettivi finiti. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 43–51.

The present paper deals with sets k_q of k points of a finite plane of order q (i.e., of a projective plane having $q+1$ points on each of its lines), subject to the condition that no three of the k points lie on a line. Let h denote the minimum positive number of chords of k_q issuing from a point of the plane (situated on some chord of k_q), and put

$$d = (k-1)(k-2)/2 - hq;$$

then it is shown that, in order that k_q be complete, i.e., not contained in a $(k+1)_q$, it is necessary that $d \geqq 0$. This condition is far from being sufficient, since, e.g., no k_q can be complete in the following cases (where $\lambda = q-k$ is necessarily $\geqq -2$ for even q, and $\geqq -1$ for odd q):

k even:

$$\begin{aligned} &k > 2(h^2-h+1), && d = 0, && h > 1;\\ &k > 6, && d = 1, && h = 1;\\ &k > 2h(h+1), && d = 1, && h > 1; \end{aligned}$$

k odd:

$$d \leqq \frac{\sqrt{3}-1}{2}k - \frac{\sqrt{3}+1}{2\sqrt{3}}k + \frac{2-\sqrt{3}}{2\sqrt{3}};$$

$$k > \lambda(\lambda+3)(\lambda^2+\lambda+1), \quad d = \tfrac{1}{2}(\lambda+1)(\lambda+2);$$

$$k < 2(h+1)+\sqrt{(2h+1)}, \quad d = \tfrac{1}{2}(k-2h-1).$$

B. Segre (Rome)

6563:

★Fischer, William L. Kritik der Nicht-Euklid'schen Geometrie. Uni-Druck, Munich, 1959. ii + 103 pp.

Das Buch besteht aus den folgenden fünf Kapiteln und einem Anhang: Widerlegung der nicht-euklid'schen Geometrie aus Lehrsätzen der absoluten Geometrie; die Denkfehler und Sophistereien der NEG-Erfinder; über die Vollkommenheit der Euklid'schen Geometrie; einige Nebenprobleme; Ausblick.

CONVEX SETS AND GEOMETRIC INEQUALITIES

6564:

Gergely, Eugen. La généralisation de la théorie polaire sur les ovales et les ovaloïdes. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 143–160. (Romanian. Russian and French summaries)

A study of polar curves and conjugate curves of ovals and similarly for ovaloids. For example, it is shown that through every interior point of an oval there is a single polar curve in each direction. There are 5 theorems, many constructions, and 19 figures.

E. R. Lorch (New York, N.Y.)

6565:

Leichtweiss, Kurt. Über die affine Exzentrizität konvexer Körper. Arch. Math. **10** (1959), 187–199.

Let B be a closed bounded convex set in n-dimensional euclidean space, $R(B)$ and $r(B)$ the radii of the smallest circumscribed and the greatest inscribed sphere, $D(B)$ the diameter and $\Delta(B)$ the width (Dicke), and let $\alpha(K_n) = \inf R(B)/r(B)$, $\beta(K_n) = \inf D(B)/\Delta(B)$, where B ranges over a class K_n of all sets which can be derived from a given set by affine transformations. $\alpha(K_n)$ and $\beta(K_n)$ may be

considered as measures of the affine eccentricity of the sets $B \in K_n$. The author proves the following. (I) $1 \leqq \alpha(K_n) \leqq n$; every value of α in this interval is taken by some class K_n; $\alpha = 1$ if and only if K_n is the class of ellipsoids; $\alpha = n$ if and only if K_n is the class of simplexes. (II) $1 \leqq \beta(K_n) \leqq \sqrt{n}$, where again every value of β in this interval is taken by some class K_n; $\beta(K_n) = 1$ if and only if K_n contains a set of constant width; $\beta(K_2) = \sqrt{2}$ if and only if K_2 is the class of parallelograms; $\beta(K_3) = \sqrt{3}$ if and only if K_3 contains either the regular hexahedron or the regular octahedron; but for $n > 3$ the class of all parallelepipeds and the class of their polar reciprocals are not the only classes for which $\beta(K_n) = \sqrt{n}$. Special cases of these and some further results were proved earlier by the reviewer [Math. Ann. **113** (1937), 713–747; **115** (1938), 379–411], O. B. Ader [Duke Math. J. **4** (1938), 291–299], F. John [Univ. Kentucky Research Club Bull. **8** (1942), 8–11; MR **4**, 252], but the methods of proof in the present paper differ from the earlier ones.

F. A. Behrend (Melbourne)

6566:

Hartman, Philip; and Valentine, Frederick A. On generalized ellipses. Duke Math. J. **26** (1959), 373–385.

Let $B(S)$ denote the boundary of the set S and S_p the translate of S by a point p in euclidean space E_n (i.e., $S_p = S + p$, where $+$ means vectorial addition). For a compact convex set K, let $W_i^n(K)$ be the mixed volume of K taken $n-i$ times and the unit sphere taken i times [see Bonnesen and Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934; pp. 28–49]. The authors obtain the following results: If R and S are compact convex sets in E_n and $K(p)$ denotes the convex hull of $R \cup S_p$, then $W_i^n(K(p))$ is a convex function of p in E_n for $i = 0, 1, 2, \cdots, n-1$. If $n > 1$ and S_p reduces to the point p, then: (a) $V(p) = W_0^n(K(p))$ is strictly convex on a line segment if and only if for every pair of points p_0, p_1 on the segment there exists a tangent hyperplane of $B(R)$ which strictly separates p_0 and p_1; (b) if the supporting function of R is analytic or of class C^m $(0 \leqq m \leqq \infty)$ the same holds for $V(p)$, $W_{n-2}^n(K(p))$ and $W_{n-1}^n(K(p))$ on the exterior of R. For the plane, $n = 2$, these results conduce to the following generalization of the familiar construction of the ellipse by means of a flexible loop and two pegs: Let $L(p)$ be the length of the boundary of $K(p)$ and λ the length of $B(R)$; then the set of points C: $L(p) \leqq c$ $(c > \lambda)$ is a smooth strictly convex set C and for each point p on the boundary of C the tangent line to C at p makes equal angles with the generators of $K(p)$.

L. A. Santaló (Buenos Aires)

6567:

Santaló, L. A. A new affine invariant of plane and solid convex bodies. Math. Notae **16** (1957/58), 78–91. (Spanish)

Let Q be a plane bounded convex domain. Let $\Delta = \Delta(\varphi) > 0$ denote the distance between the two parallel supporting straight lines of Q in the direction φ. Then $J = \int_0^\pi \Delta^{-2} d\varphi$ is invariant under unimodular affinities. If F is the area of Q, then $JF \leqq \pi^2/4$, equality holding if and only if Q is an ellipse. The author conjectures $JF \geqq 3/2$, equality characterizing the triangles. He proves $JF > 1$. If the boundary of Q has a continuous curvature $\kappa = \kappa(\varphi)$, the numbers

$$J_m = \int_0^{2\pi} (\Delta^{-1}\kappa^{1/3})^m \cdot \kappa^{-2/3} d\varphi \quad (m = 0, \pm 1, \pm 2, \cdots)$$

are invariant under the same group.

The corresponding results are obtained in 3-space and—except for J_m—in n-space.

P. Scherk (Toronto, Ont.)

6568:

Melzak, Z. A. Minkowski's theorem with curvature limitations. I. Canad. Math. Bull. **2** (1959), 151–158.

The object of the research begun in this paper is to investigate how far the constant 4 can be improved in Minkowski's fundamental theorem on lattice points in a plane convex body K, if a curvature limitation is imposed on K. Only the integer lattice is considered. The convex bodies considered are the "r-regions", with the property that through each boundary point there passes a circle of radius r which contains K completely.

The paper, stated to be preliminary to further work, consists of qualitative results about maximal r-regions. A generalization to n dimensions is considered, and it is shown that a maximal r-region in n-space is the intersection of at most $f(n)$ solid spheres, where $f(n)$ is an even integer and $f(n) < 2^n n(n-2)! + n - 2/(n-1)$.

A. M. Macbeath (Dundee)

6569:

Florian, A. Zum Problem der Überdeckung einer Kugel durch Kugeln. Monatsh. Math. **63** (1959), 351–355.

The author considers the problem of finding the radius r of the largest (solid) sphere that can be covered by four spheres with given radii r_i $(i = 1, 2, 3, 4)$. He shows that r is larger than the largest of the r_i's if and only if there is a tetrahedron whose circumcentre is in its interior and whose faces have the r_i's as circumradii and have no obtuse angles: under this condition the smallest spheres circumscribing the faces together cover a larger sphere (of radius r).

H. P. Mulholland (Exeter)

6570:

Barbilian, D. Sur un principe de métrisation. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 69–116. (Romanian. Russian and French summaries)

In an earlier paper [Časopis Pěst. Mat. Fys. **64** (1934/35), 182–183] the author introduced and stated without proof some properties of the metric space obtained by attaching to each two points a, b of the interior of a simple closed plane curve K the distance

$$d(a, b) = \max_{p,q \in K} \ln [(pa/pb)(qb/qa)].$$

The present paper greatly extends this metrization procedure and investigates in detail the resulting space. Let K denote a closed subset of a topological space T, J any subset of T and (PA) a real positive function of points P, A $(P \in K, A \in J)$ which is continuous with respect to P for A fixed. Suppose the ratio $(PA)/(PB)$ is constant only when $A = B$ (P a variable element of K and A, B fixed elements of J). Putting $M = \max_{P \in K} (PA)/(PB)$, $m = \min_{P \in K} (PA)/(PB)$, define $d(A, B) = \ln M/m$ $(A, B \in J)$. The paper studies the geometry of the resulting metric space.

L. M. Blumenthal (Columbia, Mo.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 6336, 6347, 6426, 6460, 6463, 6528, 6581.

6571:

Semadeni, Z. Sur les ensembles clairsemés. Rozprawy Mat. **19**, 39 pp. (1959).

This paper is a careful study of scattered sets, i.e., sets without non-empty dense-in-itself subset. Several characterizations of scattered sets are collected here. Besides the classical definitions of Cantor, Young, Denjoy, Mazurkiewicz-Sierpiński, Fréchet, Lusin, one finds characterizations such as the following. A metrizable space X is scattered if and only if it has one of the properties: (1) no set $A \subset X$ is homeomorphic to the set of rational numbers; (2) every countable set $A \subset X$ is scattered; (3) X is an absolute G_δ such that card A = card $\bar{A}$ for every $A \subset X$. According to a theorem of B. Knaster and K. Urbanik [Fund. Math. **40** (1953), 194–202; MR **15**, 641], every scattered separable metric space is homeomorphic to a subset of a scattered compact metric space. Generalizations of this theorem are obtained. Among others, it is shown that every scattered locally separable metric space is homeomorphic to a dense subset of a scattered locally compact metric space. It is also proved that every scattered compact Hausdorff space satisfying the first countability axiom is metrizable. There is a collection of 18 examples of scattered spaces, each with certain specified properties. The paper concludes with a list of 15 unsolved questions, in particular, the problem of topologically embedding a scattered completely regular space in a scattered compact Hausdorff space. [See also #6528.]

Ky Fan (Notre Dame, Ind.)

6572:

Knaster, B.; Lelek, A.; et Mycielski Jan. Sur les décompositions d'ensembles connexes. Colloq. Math. **6** (1958), 227–246.

Various examples of decompositions of a connected set X into a denumerable collection of closed (in X) sets have been given previously. However, every example given is not locally connected. The authors give a simple example of a connected, hereditarily locally connected X which is the denumerable union of disjoint arcs, each composed of a finite number of semi-circumferences. A plane set X is also given which is connected, locally connected (but not hereditarily locally connected) and is the union of denumerably many disjoint segments of lines. The following theorems are proved (X designates a connected metric space). Theorem 1: Let $X = \sum_{i=1}^{\infty} X_i$ be a series of disjoint, non-empty, closed (in X) sets such that $\lim_{i\to\infty}$ diameter $X_i = 0$. Then $X = M + N$, where M and N are disjoint closed sets such that $M = \sum_{j=1}^{\infty} X_{i_j}$, $N = \sum_{j=1}^{\infty} X_{k_j}$. Theorem 2: Let $X = \sum_{i=1}^{\infty} X_i$ be a series of closed (in X) sets which are non-empty and disjoint. Then X is not both locally connected and complete.

In addition, some problems are proposed.

Hing Tong (Middletown, Conn.)

6573:

Kuratowski, K. Sur les composantes de l'espace des transformations d'un espace localement compact en un rétracte de voisinage. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 565–571.

Let X and Y be separable metric spaces such that X is locally compact and Y is a neighborhood retract (not necessarily compact). Let Y^X designate the space of continuous mappings of X into Y (by convergence of elements of Y^X is meant continuous convergence). Let $L(Y^X)$ designate the family of components of Y^X. Let $\Gamma, \Gamma_1, \Gamma_2, \cdots$ be components. Define $\Gamma = \operatorname{Lim} \Gamma_k$ when there exists $f, f_1, f_2, \cdots$ such that $f_k \in \Gamma_k$ ($k = 1, 2, 3, \cdots$) and $\lim f_k = f \in \Gamma$. $L(Y^X)$ thus becomes a topological space. It is shown that this space is homeomorphic to a closed subset of the space of irrationals. Besides this theorem some remarks and additional results are also given.

Hing Tong (Middletown, Conn.)

6574:

Kuratowski, K. Un critère de coupure de l'espace euclidien par un sous-ensemble arbitraire. Math. Z. **72** (1959/60), 88–94.

Let X be a subset of the n-sphere S_n. It is a well-known theorem of Borsuk that if X is compact then $S_n - X$ is connected if and only if the function space S_{n-1}^X is connected. This result is shown to hold for arbitrary subsets X of S_n provided the topology of S_{n-1}^X is defined as the inverse limit of S_{n-1}^F where F ranges over the compact subsets of X. A more detailed quantitative result is stated without proof.

S. Eilenberg (New York, N.Y.)

6575:

Davison, Walter F. Mosaics of compact metric spaces. Trans. Amer. Math. Soc. **91** (1959), 525–546.

The spaces considered here are roughly those in which the topology is determined by the compact metrisable subspaces. (If, instead, arbitrary compact subspaces are allowed, such spaces have been considered by D. E. Cohen [Quart. J. Math. Oxford Ser. (2) **5** (1954), 77–80; MR **16**, 62] and K. Morita [Proc. Japan Acad. **32** (1956), 544–548; MR **19**, 49].) More precisely, let $X = \bigcup X_a$ ($a \in A$), where each X_a has a compact metric topology, and suppose that for all $a, b \in A$, and for each set E closed in X_a, $E \cap X_b$ is closed in X_b. Taking the closed sets F of X to be those for which $F \cap X_a$ is closed in X_a for all $a \in A$, X is the "mosaic space" of the "mosaic" X_a; each X_a is a closed subspace of X. All mosaic spaces X are obtainable by taking the X_a's to consist of either all the compact metrisable subspaces of X, or the convergent sequences plus their limits. They coincide with the *-convergence classes (i.e., topological spaces in which the closed sets are determined by the convergent sequences) in which limits of convergent sequences are unique. A first countable space is mosaic if and only if it is Hausdorff. Every quasicompact image (= decomposition space) of a mosaic space is mosaic, and conversely every mosaic space is a quasicompact image of a locally compact metric space (an analogue of a theorem of Morita [loc. cit.] and of Cohen [Proc. London Math. Soc. (3) **7** (1957), 219–248; MR **19**, 441]). Every open, or closed, subspace of a mosaic space is mosaic; for every subspace to be mosaic it is necessary and sufficient that closures be determined by convergent sequences. If X' is the 1-point countable compactification of X, X is mosaic if and only if X' is; X is imbeddable in a compact mosaic space if and only if X' is compact, or equivalently if all countably compact subsets of X are compact. For the product of a fixed mosaic space X with every mosaic space Y to be mosaic, it is necessary and

sufficient that X be locally countably compact and regular. Further results include a generalisation to "functional mosaics" specified by mappings of mosaic spaces, results on continuity and mapping spaces, and examples (some of independent interest) showing the need for the hypotheses in the theorems. {However, the example in 3.9, said to answer a question of the reviewer about the metrisability of decomposition spaces, has answered a different question; the reviewer wished to dispense, not with the "frontier hypothesis", but with the hypothesis that the image space be paracompact. For the mapping space results, reference should be made to Morita, Proc. Japan Acad. **32** (1956), 671–675 [MR **19**, 49]. On p. 526 line 2, Y is presumably a misprint for X.}

A. H. Stone (Manchester)

6576:

Burgess, C. E. Some conditions under which a homogeneous continuum is a simple closed curve. Proc. Amer. Math. Soc. **10** (1959), 613–615.

If M is a nondegenerate compact metric continuum, which is not a triod (i.e., if T is a subcontinuum of M, $M-T$ is not the sum of three mutually separate point sets), any two of the following conditions are equivalent. (1) M is a simple closed curve; (2) M is locally connected and nearly homogeneous; (3) M is decomposable and nearly 2-homogeneous; (4) M is 2-homogeneous. The author also characterizes a simple closed curve as a nondegenerate compact continuum which can be homogeneously embedded in the plane.

In the terminology of this paper, n-homogeneity refers to sets of n points and not to n-tuples.

F. B. Jones (Chapel Hill, N.C.)

6577:

Ponomarev, V. Open mappings of normal spaces. Dokl. Akad. Nauk SSSR **126** (1959), 716–718. (Russian)

Let X, Y be normal spaces. If a multivalued mapping f of X onto Y is perfect strongly continuous open [for these notions, see the author's note in same Dokl. **124** (1959), 268–271; MR **21** #1582], then the (unique) strongly continuous extension of f to βX is open (the proof is based on a strengthened form of two results, loc. cit.). Corollary: if f is single-valued continuous, open and closed, and the $f^{-1}(y)$ are compact, then the extension of f to βX is an open mapping. Moreover, it is proved that, under the same assumptions on a single-valued f, Y is a Q-space whenever X is so.

M. Katětov (Prague)

6578:

Whyburn, G. T. Mapping norms. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1431–1436.

If X and Y are metric spaces and f a mapping of X into Y, the norm $N(f)=\sup \delta(f^{-1}(y))$, for $y \in Y$, where $\delta(A)$ is the diameter of A. Let $\mathfrak{F}$ be a class of mappings of X. The author considers conditions for which there exists a real number $d(X, \mathfrak{F})>0$ such that $N(f) \geq d(X, \mathfrak{F})$ for all $f \in \mathfrak{F}$. The case where X is a simple closed curve J is studied and the results applied to obtain a theorem on 2-manifolds.

If J has metric ρ, a, b, c are distinct points of J, and $\delta(a, b, c)$ the minimum of $\rho(a, bc)$, $\rho(b, ca)$, $\rho(c, ab)$, where bc is the arc of J not containing a, etc., let $\alpha(J)=\sup \delta(a, b, c)$. Theorem: If f: $J \to J$ is open and non-topological, then $N(f) \geq \alpha(J)$. Theorem: If f is real-valued on J, then $N(f) \geq \alpha(J)$. Theorem: If J is in the plane and contains in its interior a disk of radius r and if f is open and non-topological, then $N(f) \geq r\sqrt{3}$. These and other results imply the theorem: If M is a 2-manifold there is a $d(M)>0$ such that if f: $M \to M$ is light, open, and of degree >1 on a dense set of M, then $N(f) \geq d(M)$.

M. E. Shanks (Lafayette, Ind.)

6579:

Brown, Morton. Weak n-homogeneity implies weak $(n-1)$-homogeneity. Proc. Amer. Math. Soc. **10** (1959), 644–647.

The author proves the deceptively trivial looking proposition of the title by a decidedly non-trivial combinatorial argument. In fact, the proposition is a corollary of the following theorem: Suppose that G is a group of 1-1 transformations of a set S onto itself which is weakly n-transitive over an infinite subset S of S (i.e., if each of the subsets X and Y of S has exactly n elements, there is an element g of G such that $g(X)=Y$). Then G is weakly $(n-1)$-transitive over S. In the finite case when S has exactly k elements ($k \geq n$), the proposition is still valid provided n and $C_{k,n-1}$ are relatively prime. After showing that this latter condition may be satisfied for some $k \geq 2n$, the author deduces the infinite case from the finite case.

With respect to homogeneity, the author's terminology is not universal since he makes it conform to the concepts of transitivity and weak transitivity of groups operating on a set. Thus n-homogeneity refers to ordered n-tuples, while weak n-homogeneity refers to sets.

F. B. Jones (Chapel Hill, N.C.)

6580:

Derwent, J. E. On the covering homotopy theorem. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 275–279.

The author's purpose is to give a simplified proof of W. Huebsch's generalisation [Ann. of Math. (2) **61** (1955), 555–563; MR **19**, 974] of the Hurewicz-Steenrod covering homotopy theorem. Huebsch's generalisation consisted in allowing the base-space to be paracompact. The simplification of technique in the present paper is made possible by the use of stronger theorems about paracompact spaces.

J. F. Adams (Cambridge, England)

ALGEBRAIC TOPOLOGY

See also 6580, 6594, 6758, 6949, 7103.

6581:

Švedov, I. Proof of a theorem on the homeomorphism of polyhedra and point sets. Dokl. Akad. Nauk SSSR **122** (1958), 566–569. (Russian)

In this note, the author discusses several types of spectra, in the sense of P. S. Aleksandrov [see same Dokl. **97** (1954), 757–760; Trudy Moskov. Mat. Obšč. **4** (1955), 405–420; MR **17**, 69]. He completes and simplifies a theorem of Aleksandrov, which gives necessary and sufficient conditions, in terms of spectra, for two sets in Euclidean spaces to be homeomorphic.

D. W. Kahn (Cambridge, Mass.)

6582:

Wu, Wen-Tsün. On the relations between Smith operations and Steenrod powers. Acta Math. Sinica **7** (1957), 235–241. (Chinese. English summary)

Let K be a complex, I_p the field of integers mod p (p a prime). The Steenrod powers

$$\mathrm{St}^k_{(p)} = \mathrm{St}^k : H^r(K, I_p) \to H^{r+k}(K, I_p)$$

were discovered by Steenrod [*Reduced powers of cohomology classes*, Cours de Collège de France, Paris, 1951] by considering $K^p = K \times \cdots \times K$ (p times) under the cyclic transformation $t(x_1, \cdots, x_p) = (x_p, x_1, \cdots, x_{p-1})$, $x_i \in |K|$. Now according to P. A. Smith [Lefschetz, *Algebraic topology*, Amer. Math. Soc., New York, 1942; MR **4**, 84; pp. 350–373; Richardson and Smith, Ann. of Math. (2) **39** (1938), 611–633] under t, one may introduce naturally a system of homomorphisms

$$\mathrm{Sm}^{(p)}_k = \mathrm{Sm}_k : H_r(K, I_p) \to H_{r-k}(K, I_p).$$

The author discovered [Colloque de Topologie de Strasbourg, 1951, no. IX, Biblioth. Nat. Univ. Strasbourg, 1952; MR **14**, 491] that these two systems of operation were equivalent. Each system determines the other. Thus the Steenrod powers are given a more natural and simpler definition; such powers are made directly related to the Smith theory. However, in the paper cited above, the author had to appeal to Thom's intrinsic axiomatic theory of Steenrod powers [R. Thom, ibid., no. V; MR **14**, 492]. In this paper, a direct proof (without using Thom's work) is given. It is shown that

$$\sum_{j=0}^{k} (-1)^j \mathrm{Sm}^{k-j}\,\mathrm{St}^j = \begin{cases} 0, & k>0 \\ I, & k=0 \end{cases} \quad (p=2 \text{ or } pk \text{ is odd});$$

$$\sum_{j=0}^{k} \mathrm{Sm}^{2k-2j}\,\mathrm{St}^{2j} = \begin{cases} 0, & k>0 \\ I, & k=0 \end{cases} \quad (p>2).$$

It is also proved that

$$\sum_{j=0}^{k} (-1)^j \mathrm{St}^{k-j}\,\mathrm{Sm}^j = \begin{cases} 0, & k>0 \\ I, & k=0 \end{cases} \quad (p=2, \text{ or } pk \text{ is odd});$$

$$\sum_{j=0}^{k} \mathrm{St}^{2k-2j}\,\mathrm{Sm}^{2j} = \begin{cases} 0, & k>0 \\ I, & k=0 \end{cases} \quad (p>2).$$

Hing Tong (Middletown, Conn.)

6583:

Mizuno, Katuhiko. Semi cubical theory on homotopy classification. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 43–62.

In this paper, the author presents a reworking of the Postnikov theory [Dokl. Akad. Nauk SSSR **76** (1951), 359–362; MR **13**, 374]. He is particularly concerned to give an explicit discussion of the question of the existence of a homotopy between two maps; for this reason he works in the cadre of semi-cubical complexes. The paper finishes by showing how the results apply to the case in which the relevant Postnikov invariants are additive.

J. F. Adams (Cambridge, England)

6584:

Spanier, E. H. Function spaces and duality. Ann. of Math. (2) **70** (1959), 338–378.

Let A, B be spaces with base point and let $A \hat{\times} B$ be their reduced ('smashed') product. There is then defined a slant product $y/z \in H^{n-q}(B)$, where $y \in H^n(A \hat{\times} B)$, $z \in H_q(A)$. Given a map $u : X' \hat{\times} X \to S^n$, define $\phi_u : H_q(X') \to H^{n-q}(X)$ by $\phi_u(z) = u^*(s_n)/z$, where s_n is the orienting generator of $H^n(S^n)$. The author describes the map u as a duality map, and X' as an n-dual of X if ϕ_u is an isomorphism for all q. This notion generalizes that defined by Spanier and Whitehead [Mathematika **2** (1955), 56–80; MR **17**, 653], and amends it by a dimension shift. For if X, X' are embedded in S^{n+1} in such a way that X' is a deformation retract of the complement of X then a canonical map $u : X' \hat{\times} X \to S^n$ may be defined which is a duality map in the above sense. The burden of the present paper—well sustained—is that the duality relations in S-theory hold in this more general formulation and that the definitions and properties of the basic concepts are thereby more conveniently and naturally adduced.

After two sections devoted to standard preliminaries and the properties of function spaces the author introduces the concept of a direct spectrum. This is a sequence $\mathbf{X} = (X_k, \rho_k)$ of based spaces and maps $\rho_k : SX_k \to X_{k+1}$, $k = 0, 1, \cdots$, such that (3.1) there exists Q such that X_k is $(Q+k)$-connected, $k \geq 0$, and (3.2) to each q there exists N_q such that

$$\rho_{k*} : H_{q+k+1}(SX_k) \xrightarrow{\sim} H_{q+k+1}(X_{k+1})$$

if $k \geq N_q$. A map of spectra is defined in the obvious way. The homology groups of $\mathbf{X}$ are defined by $H_q(\mathbf{X}) = \varinjlim H_{q+k}(X_k)$; in the light of (3.2), $H_q(\mathbf{X}) \approx H_{q+k}(X_k)$ if $k \geq N_q$. A map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is called a weak equivalence if it induces homology isomorphisms. Of particular interest are the spectrum $\mathbf{S}(X)$, the suspension spectrum of X, and the spectrum $\mathbf{F}(X) = (F(X, S^k), \lambda_k)$. Here (and subsequently) X is a connected polyhedron, $F(X, S^k)$ is the function space of maps $X \to S^k$ and $\lambda_k : SF(X, S^k) \to F(X, S^{k+1})$ is given by $(\lambda_k(f, t))(x) = (fx, t)$. An important lemma (4.2) establishes a natural isomorphism $\phi : H_q(\mathbf{F}(X)) \approx H^{-q}(X)$, so that a duality map $u : X' \hat{\times} X \to S^n$ induces a weak equivalence $\mathbf{g} : \mathbf{S}(X') \to S^n\mathbf{F}(X)$: this shows the essential uniqueness of the n-dual and justifies the description of $\mathbf{F}(X)$ as the functional dual of X. There is an evident isomorphism of the S-groups $\{Y, S^n\mathbf{F}(X)\}$ and $\{Y \hat{\times} X, S^n\}$, whence a natural isomorphism

$$D : \{Y, S^n\mathbf{F}(X)\} \approx \{X, S^n\mathbf{F}(Y)\}.$$

Combining $D : \{X, S^n\mathbf{F}(Y')\} \approx \{Y', S^n\mathbf{F}(X)\}$ with the weak equivalences $\mathbf{S}(X') \to S^n\mathbf{F}(X)$, $\mathbf{S}(Y) \to S^n\mathbf{F}(Y')$ induced by duality maps $u : X' \hat{\times} X \to S^n$, $v : Y' \hat{\times} Y \to S^n$, we obtain the fundamental isomorphism (5.9)

$$D_n(u, v) : \{X, Y\} \approx \{Y', X'\}.$$

A rule is given for recognizing whether a given element of $\{X, Y\}$ and a given element of $\{Y', X'\}$ are paired under D_n.

Section 6 deals with the naturality properties of the duality. All goes smoothly modulo bothersome signs incurred in commuting suspensions with smashed products. In an evident way two duality maps $u : X' \hat{\times} X \to S^n$, $v : Y' \hat{\times} Y \to S^n$ combine to give a duality map $u \oplus v : (X' \vee Y') \hat{\times} (X \vee Y) \to S^n$; and two duality maps $u : X' \hat{\times} X \to S^n$, $v : Y' \hat{\times} Y \to S^m$ combine to give a duality map $u \otimes v : (X' \hat{\times} Y') \hat{\times} (X \hat{\times} Y) \to S^{n+m}$. If $u : X' \hat{\times} X \to S^n$, $v : Y' \hat{\times} Y \to S^n$ are duality maps and $f : X \to Y$, $f' : Y' \to X'$

are such that $u \circ (f' \divideontimes 1) \simeq v \circ (1 \divideontimes f) : Y' \divideontimes X \to S^n$, then the mapping cones of f and f' are $(n+1)$-dual. There is an isomorphism $\Gamma : \{X, Z \divideontimes Y'\} \approx \{X \divideontimes Y, S^n Z\}$ if X, Z are polyhedra and Y, Y' are n-dual and this suggests (by taking Y to be an appropriate Moore space), two equivalent definitions of S-groups with coefficients.

The final section indicates how the theory may be relativized.

{The reviewer noticed several misprints; in particular, on l. 4 of the final paragraph of p. 350, 'dim X' should be '2 dim X', and in the statement of Theorem 5.5 '$S^k\mathbf{F}(X)$' should be '$S^n\mathbf{F}(X)$'.} *P. J. Hilton* (Birmingham)

6585:

Kervaire, Michel A. A note on obstructions and characteristic classes. Amer. J. Math. **81** (1959), 773–784.

Let G be one of the classical Lie groups, $\mathrm{U}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(n)$. Let ξ be a stable principal G-bundle over a finite dimensional CW-complex K (stability means that the homotopy groups $\pi_{q-1}(G)$ are stable for $q \leq$ dimension K). Assume that ξ admits a cross section f over the $(q-1)$-skeleton of K. Take q to be even, $q=2r$, if $G=\mathrm{U}(n)$, and take q divisible by 4, $q=4k$, if $G=\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$. Then $\pi_{q-1}(G)$ is infinite cyclic in all cases by the results of R. Bott [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 933–935; MR **21** #1588]. The obstruction class $o(\xi, f) \in H^q(K, \pi_{q-1}(G))$ to extending f over the q-skeleton can be regarded up to a sign as an integer class. Denote by $c_r(\xi)$, $p_k(\xi)$, $e_k(\xi)$ the Chern class, Pontrjagin class, or symplectic Pontrjagin class of ξ according as $G=\mathrm{U}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(n)$, respectively. Then the basic result of this note is contained in the following three formulae: (i) $c_r(\xi) = (r-1)!\,o(\xi, f)$ if $G=\mathrm{U}(n)$; (ii) $p_k(\xi) = (2k-1)!\,a_k \cdot o(\xi, f)$ if $G=\mathrm{SO}(n)$; and (iii) $e_k(\xi) = (2k-1)!\,b_k \cdot o(\xi, f)$ if $G=\mathrm{Sp}(n)$. In these formulae, $a_k=1$ for k even and $a_k=2$ for k odd, while $b_k = 2/a_k$.

As a corollary, the author derives another proof of a recent theorem of F. Peterson [Ann. of Math. (2) **69** (1959), 414–421; MR **21** #1593]. He also uses these formulae to determine the Postnikov invariants of the classifying spaces of $\mathrm{U}(n)$, $\mathrm{SO}(n)$, and $\mathrm{Sp}(n)$ in the stable range.

W. S. Massey (Providence, R.I.)

6586:

Wall, C. T. C. Note on the cobordism ring. Bull. Amer. Math. Soc. **65** (1959), 329–331.

Let $\mathfrak{N}$ and Ω denote the non-oriented and oriented cobordism rings of Thom and let $r : \Omega \to \mathfrak{N}$ be the natural map. Completing the results of Thom, Rohlin, and Milnor on the structure of Ω, the author now shows that there are no elements in Ω of order 4 (thus, since there are no elements of order p where p is an odd prime, all torsion is of order 2), that two oriented manifolds are cobordant if and only if all their Pontrjagin and Stiefel numbers are the same (confirming a conjecture of Thom), and finally that there is an explicitly given polynomial subalgebra $\mathfrak{B}$ of $\mathfrak{N}$ which contains $r(\Omega)$ and a map $\partial : \mathfrak{B} \to \Omega$ such that the following triangle is exact:

$$\begin{array}{ccc} \Omega & \xrightarrow{2} & \Omega \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle r} & \\ & \mathfrak{B} & \end{array}$$

Since it was already known that the torsion-free part of Ω is a polynomial algebra, that $\Omega \otimes Q$ is a polynomial algebra generated by the even-dimensional complex projective spaces $P_{2n}(C)$ and as mentioned above that Ω has no odd torsion, these results completely determine Ω.

The basic idea in the proof is the following. Let $\mathfrak{B} \subset \mathfrak{N}$ be the set of classes which contain a manifold M whose first Stiefel-Whitney class W_1 is the restriction of an integral class. If this condition is satisfied one can find a regular map $f : M \to S^1$ $(= K(Z, 1))$ which corresponds to such a cohomology class. Let $V = f^{-1}(0)$. Then V is a submanifold of M of codemension 1. The Stiefel numbers of V are determined by those of M, as is easily checked. Hence the class of V in $\mathfrak{N}$ is determined by that of M. Define $\partial' : \mathfrak{B} \to \mathfrak{N}$ by $\partial'\{M\} = \{V\}$. Next it is shown that $\mathfrak{B}$ is a polynomial algebra and generators are given. The map $\partial : \mathfrak{B} \to \Omega$ is obtained from ∂' by proving the conjecture of Thom mentioned earlier and then observing that, if V is obtained as above, then $2V$ bounds orientably and hence all Pontrjagin numbers of V are zero.

N. Stein (New York, N.Y.)

6587:

Weier, Joseph. Eine Eigenschaft des topologischen Schnittringes. Proc. Japan Acad. **35** (1959), 155–157.

Dividing the original set of a point into homotopy classes is a well-known tool in the study of transformations of manifolds of the same dimension. The author introduces this definition in the case of different dimensions, though one can doubt whether that definition is still workable in this case. No invariance theorem is pronounced, and no application is given.

H. Freudenthal (Utrecht)

6588:

Heller, Alex. A note on spaces with operators. Illinois J. Math. **3** (1959), 98–100.

Soit Π un groupe de la forme $(Z_p)^r$ opérant sans point fixe sur un espace connexe $\mathfrak{X}$. On suppose en outre que les $H_i(\mathfrak{X}, Z_p)$ sont finis pour tout i et que, pour i suffisamment grand, ils sont nuls ainsi que les $H_i(\mathfrak{X}/\Pi, Z_p)$. On désigne alors par b_i le i-ème nombre de Betti modulo p de l'espace $\mathfrak{X}$ et l'on démontre par l'homologie algébrique des inégalités de la forme

$$(k+1, r) \leq \sum_{j=1}^{k} (k-j, r) b_j + \sum_{j=k+1}^{r} (j-k-1, r) b_j$$

où $(m, n) = (m+n)!/m!n!$. De là résultent des généralisations de résultats de P. E. Conner [Ann. of Math. (2) **66** (1957), 586–588; MR **20** #2725] et P. A. Smith [Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 105–108; MR **5**, 274] sur l'existence de points fixes lorsque $\Pi = (Z_p)^r$ [resp. $(Z_p)^2$] opère sur un espace ayant même homologie que $(S^n)^{r-1}$ [resp. $S^n \times S^m$]. *P. Dedecker* (Liège)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 6758, 7052.

6589:

Brauner, H. Eine Verallgemeinerung des Problems der Cesàrokurven. Math. Ann. **138** (1959), 27–41.

A straight line g rigidly attached to the moving trihedron of a skew curve c generates a ruled surface Θ. If the parameter of distribution d of Θ is constant, g is called

a d-line. Thus g is a 0-line if and only if Θ is developable. For every d there are two isotropic d-lines collinear with and normal to the tangent. The straight lines parallel to the tangent in the rectifying plane are 0-lines. Any other d-line is called proper. A curve c with proper d-lines is a Cesàro curve, i.e. the curvature κ and the torsion τ of c satisfy a quadratic equation

$$(1) \qquad A\kappa^2+B\tau^2+C\kappa\tau = P\kappa+Q\tau$$

with constant coefficients.

Assume (1) and put $\Delta=AQ^2+BP^2-CPQ$. (i) If $\Delta P\neq 0$, the set of all the proper d-lines is a rational ruled surface. The principal normal of c is an axis of symmetry. If d is given and if d and the coefficients of (1) satisfy certain conditions, no proper d-line exists. (ii) Let $\Delta\neq 0$, $P=0$. For every $d\neq 0$ there are two proper d-lines symmetric to the principal normal and parallel to the osculating plane. They generate a ruled conoid of degree four. In certain special cases more proper d-lines exist. If $AB\neq 0$ there are no proper 0-lines. (iii) If $\Delta=0$, c is either a general helix or a Bertrand curve. In either subcase there are proper d-lines for each d.—Isotropic proper d-lines exist in the case (iii) and only then.

P. Scherk (Toronto, Ont.)

6590:

Osserman, Robert. Remarks on minimal surfaces. Comm. Pure Appl. Math. **12** (1959), 233–239.

The paper deals with some aspects of minimal surfaces, in particular regarding their conformal structure. One of the tools employed is the expression of Beltrami's second differential operator Δ for functions u defined on a surface S, given in the special form $z=f(x,y)$:

$$\Delta u = W^{-2}[(1+f_y^2)u_{xx}-2f_xf_yu_{xy}+(1+f_x^2)u_{yy}] - 2HW^{-1}[f_xu_x+f_yu_y]$$

(H = mean curvature, $W=(1+f_x^2+f_y^2)^{1/2}$); another is an inequality by E. Hopf [J. Rational Mech. Anal. **2** (1953), 519–522; MR **14**, 1119]. The following theorems are proved. (1) Let R be any region in the (x,y)-plane and let $f(x,y)$ be a non-constant function defined in R. Let H be the mean curvature of the surface $S: z=f(x,y)$. If for some positive number M we have $f(x,y)<M$ and $H\geqq 0$, then S is hyperbolic. (2) If one of the coordinate functions, say z, is harmonic on a twice-continuously differentiable surface S, then S is either a minimal surface or else a locally cylindrical surface with its generators parallel to the z-axis. (3) Let R be an arbitrary region in the (x,y)-plane, and let $z=f(x,y)$ be a function in R defining a minimal surface S. Then S is either hyperbolic or a portion of a plane. (4) Let $z=f(x,y)$ define a minimal surface S over the circle $x^2+y^2<R^2$. Then the part of S lying over the disk $D: x^2+y^2\leqq (R/2)^2$ has total curvature in absolute value $\leq c$, where c is a constant independent of R and f. (5) Let S be a simply-connected minimal surface having the property that the normals omit three distinct directions. Then S is either hyperbolic or a portion of a plane.—Simply connected minimal surfaces whose spherical image omits a whole neighborhood of a point have been investigated previously by the author [J. Math. Mech. **8** (1959), 383–385; Comm. Pure Appl. Math. **12** (1959), 229–232; MR **21** #2991, #4436], and he could prove that a complete minimal surface with these properties must be parabolic and hence a plane. *J. C. C. Nitsche* (Minneapolis, Minn.)

6591:

Özkan, Asım. Über W-(Weingarten)-Flächen, auf der die Kurven der festen Krümmung mit den Krümmungslinien 3-Sechseckwaben bilden. Rev. Fac. Sci. Univ. Istanbul. Sér. A **22** (1957), 101–122. (Turkish summary)

Let S be a W-surface in ordinary Euclidean space with principal curvatures r, r^*, mean curvature H and Gaussian curvature K. On S the family of curves defined by $r=\text{const}$, $r^*=\text{const}$, $H=\text{const}$, $K=\text{const}$, respectively, is called a family of curves with a fixed curvature. It is proved that if S has a family of plane lines of curvature, then S is either a tube surface (r or $r^*=\text{const}$) or a surface of revolution, or a surface whose family of curves with a fixed curvature and two families of the lines of curvature form a 3-hexagon honeycomb. Furthermore, a necessary and sufficient condition for a W-surface with variable mean curvature or a W-surface which is not a surface of revolution to be isothermic is that the family of the orthogonal trajectories of the curves of a fixed curvature and the two families of the lines of curvature form a 3-hexagon honeycomb. Finally, a classification is given of W-surfaces with a family of plane lines of curvature.

C. C. Hsiung (Madison, Wis.)

6592:

Arghiriade, E. Réseaux et congruences du troisième ordre. Lucrăr. Ştiinţ. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 41–54 (1959). (Romanian. French and Russian summaries)

In un lavoro precedente [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 246–250; MR **20** #4854], l'autore ha studiato "equazioni generalizzate di Laplace d'ordine (p,q)" del tipo:

$$(1) \qquad \sum_{i=0}^{p}\sum_{j=0}^{q} a_{ij}x^{ij} = 0 \quad \left(x^{ij}=\frac{\partial^{i+j}x}{\partial u^i\partial v^j}\right).$$

Nel presente lavoro l'autore, giovandosi della teoria degli spazi di Laplace, si occupa di alcuni tipi particolari di equazioni (1) con $p+q=3$. *D. Gallarati* (Genoa)

6593:

Fladt, Kuno. Über den Parallelismus von Levi-Cività in der Geometrie konstanten Krümmungsmasses. J. Reine Angew. Math. **201** (1959), 78–83.

Let R_n denote either elliptic or hyperbolic n-space. R_n can be realized as the subset

$$(x|x) = x_0^2+\varepsilon(x_1^2+x_2^2+\cdots+x_n^2) = 1 \quad (\varepsilon = \pm 1)$$

of affine $(n+1)$-space, if x and $-x$ are identified. For this model, a tangent vector is an $(n+1)$-tuple. R_n is also realized as the interior of the quadratic surface $(x|x)=0$ in projective n-space P_n ($R_n=P_n$ if $\varepsilon=+1$). A tangent vector ξ (which is taken to be an $(n+1)$-tuple from the first model) can be considered as a point in P_n. This paper is concerned with properties of a curve $\xi=\xi(s)$ in P_n, where $\xi(s)$ is the (Levi-Civita) parallel translate of a unit vector along a curve $x=x(s)$ in R_n. It is easily seen that the lines $x(s)\cdot\xi(s)$ are translates, and the author shows that $\xi(s)$ is conjugate to $x(s)$ with respect to the absolute quadratic.

The author also proves the following. Theorem 1: The tangent to the curve $\xi=\xi(s)$ at $\xi(s_0)$ is the line $x(s_0)\cdot\xi(s_0)$. Theorem 2: If $n=2$, the curves $\xi=\xi(s)$ are the (non-Euclidean) involutes of the polar curve of $x=x(s)$. Theorem

3: If $n=3$, the edge of regression of the surface enveloped by the normal planes of a curve $\xi=\xi(s)$ is the edge of regression of the polar surface of $x=x(s)$. Examples are given to illustrate Theorems 2 and 3.

L. Greenberg (Providence, R.I.)

6594:

Auslander, L.; and MacKenzie, R. E. On the topology of tangent bundles. Proc. Amer. Math. Soc. **10** (1959), 627–632.

The authors characterize the usual topology of the tangent bundle $T(M)$ of a differentiable manifold M by the following five conditions: (1) $T(M)$ is regular, (2) each point of $T(M)$ has an open neighborhood whose closure in the usual topology is compact in the usual topology, (3) the projection is continuous, (4) the relative topology induced on each fibre is the usual vector space topology, (5) if a vector field X on an open set U of M is continuous in the sense that Xf is a continuous function for every C^1 function f on U, then $X: M \to T(M)$ is a continuous cross-section. An example is given to show that the second condition is essential.

S. Kobayashi (Vancouver, B.C.)

6595:

Aczél, J. Beiträge zur Theorie der geometrischen Objekte. VI. Eindimensionale differentialgeometrische Objekte mit einer Komponente, deren Werte in beliebigen Mengen, insbesondere in zwei Intervallen liegen. Acta Math. Acad. Sci. Hungar. **10** (1959), 1–12. (Russian summary, unbound insert)

[Part V: same Acta **8** (1957), 53–64; MR **19**, 677.] It is shown that under the hypotheses of continuity and transitivity the domain of definition of a one-dimensional geometric object with one component can consist of at most two intervals. While there are no one-dimensional geometric objects with one component of order greater than three, it is shown that for each of the remaining values of the order the situation with two intervals can occur, the latter again under the hypotheses of continuity and transitivity. A classification is given.

A. Nijenhuis (Seattle, Wash.)

6596:

Kremer, Hugo Frederico. Tensor analysis in spaces with torsion. Soc. Parana. Mat. Anuário (2) **1** (1958), 38–42. (Portuguese. French summary)

"Quelques points fondamentaux d'analyse tensorielle dans les espaces de connexion non symétrique sont présentés ici. Après avoir défini la dérivation covariante, l'auteur analyse la dérivé covariante des deltas de Kronecker. Il définit ensuite la dérivée covariante des densités tensorielles, et obtient finalement les expressions de deux tenseurs de courbure pour les espaces considérés."

Résumé de l'auteur

6597:

Hicks, N. A theorem on affine connexions. Illinois J. Math. **3** (1959), 242–254.

This paper may be regarded as a sequel to two earlier papers, namely, W. Ambrose and I. M. Singer, Trans. Amer. Math. Soc. **75** (1953), 428–443 [MR **16**, 172] and W. Ambrose, Ann. of Math. (2) **64** (1956), 337–363 [MR **21** #1627].

Let M be a C^∞ manifold, $B(M)$ the bundle of bases over M, and π the projection map of $B(M)$ onto M. An affine connexion on $B(M)$ is determined by a C^∞ 1-form ω with values in the Lie algebra of the general linear group $\mathrm{GL}(d, R)$, where d is the dimension of M and R is the field of reals. It is assumed that the connexion is complete, i.e., paths can be extended indefinitely in terms of the affine parameter. A space X is defined with the property that, for a given manifold M, a complete connexion ω on M, and a "key" base $b \in B(M)$, the curvature and torsion forms on $B(M)$ induce on X certain real-valued functions K_{ij} and K_i $(i=1, 2, \cdots, d)$.

Let $'M$ be another C^∞ manifold with complete connexion $'\omega$ on $B('M)$. Then a diffeomorphism ϕ of M onto $'M$ is said to be "connexion preserving" if the induced diffeomorphism $\tilde{\phi}$ of $B(M)$ onto $B('M)$ carries $'\omega$ into ω. The main theorem proved in the paper is as follows: Let M and $'M$ be two d-dimensional simply connected C^∞ manifolds each carrying a complete affine connexion. If after choosing $b \in B(M)$ and $'b \in B('M)$, the functions $K_{ij}={}'K_{ij}$ and $K_i={}'K_i$ on X, then there is a connexion preserving diffeomorphism of M onto $'M$.

As a corollary to the theorem it follows that a simply connected manifold which carries a complete connexion whose curvature is zero and torsion invariant under parallel translation is necessarily a Lie group. An example shows this no longer holds if the simply connected hypothesis is replaced by just connectedness together with the assumption that the holonomy group is the identity.

T. J. Willmore (Liverpool)

6598:

Liang, You-dong. Characterizations of certain K^*-spaces. Acta Math. Sinica **9** (1959), 69–75. (Chinese. English summary)

A K_n^*-space is a Riemannian n-space whose curvature tensor satisfies either (1) $R_{hijk,l}=R_{hijk}\kappa_l$ or (2) $R_{hijk,l}=0$ and $R_{hijk}\kappa_l+R_{hikl}\kappa_j+R_{hilj}\kappa_k=0$, where κ_l is a non-zero vector [Proc. London Math. Soc. (2) **52** (1950), 36–64; MR **12**, 283]. The author proves that a K_n^*-space is not a flat extension of V_2 if and only if $R_{ij}=\rho\lambda_i\lambda_j$, where R_{ij} is the Ricci tensor, ρ a scalar and λ_i a null parallel vector field. The theorem leads immediately to a classification of K_n^*-spaces according to the rank r of R_{ij}, i.e., Einstein spaces when $r=0$, flat extensions of K_3^* or non-simple and non-Einstein when $r=1$ and $K_n^*=V_2\times E_{n-2}$ when $r=2$. Several results are obtained to characterize K_n^*-spaces as conformal flat spaces or harmonic spaces, two of which read as follows. A conformally flat space is a K_n^*-space if and only if it admits a null parallel vector field; a K_n^*-space admitting $n-3$ independent parallel vector fields is a harmonic space if and only if it is an Einstein space admitting at least two null parallel vector fields.

T. K. Pan (Norman, Okla.)

6599:

Nagano, Tadashi. Isometries on complex-product spaces. Tensor (N.S.) **9** (1959), 47–61.

Let M be a differentiable manifold of even dimension. By a complex-product structure (in short, a (C, P)-structure) over M, the author means a pair of tensor fields C, P of type (1, 1) such that $P^2=-C^2=1$, $CP+PC=0$. Such a structure will be called isometric if there is, in addition, a Riemannian metric over M invariant under both C and P at every point. Over some isometric (C, P)-spaces, the author defines a radiator to be a vector field

satisfying certain quite complicated conditions. Let N be a Finsler space. Then the space $T(N)-N$ of non-zero tangent vectors over N has a natural isometric (C, P)-structure with a radiator, and so the latter structure can be regarded as generalization of the Finsler metric. In this paper, the author determines the isometric (C, P)-spaces (with or without radiator) which admit an automorphism group of "high" dimension.

H. C. Wang (Evanston, Ill.)

6600:

Loewner, Charles. On some transformation semigroups invariant under Euclidean or non-Euclidean isometries. J. Math. Mech. **8** (1959), 393–409.

$\mathfrak{G}$ désignant le groupe des isométries de l'espace V^m, à m dimensions, euclidien, sphérique, ou hyperbolique, l'A. détermine les systèmes minimaux de transformations infinitésimales constituant un cône convexe, de dimension finie, invariant par $\mathfrak{G}$. En géométrie euclidienne, seul le cas $m=1$ donne des solutions autres que des translations; dans le cas de la géométrie sphérique de dimension 2, la solution est donnée au moyen de fonctions sphériques; on obtient des résultats analogues en géométrie hyperbolique.

J. Lelong (Paris)

6601:

Hsü, Chin-shui. Characterization of some elementary transformations. Proc. Amer. Math. Soc. **10** (1959), 324–328.

Let S_i $(i=1, 2)$ be closed orientable surfaces of class C^2 in E^3. Let $\mathfrak{x}_i$ be the position vector of S_i from the origin 0, and H_i the mean curvature of S_i at $\mathfrak{x}_i$. Three close analogues to a theorem of H. Hopf and K. Voss [Arch. Math. **3** (1952), 187–192; MR **14**, 583] are proved, of which the first two are the following. Let h be a differentiable homeomorphism of S_1 on S_2, where S_i does not contain pieces of cones with apex 0. (1) If the lines connecting corresponding points $\mathfrak{x}_1$ and $h(\mathfrak{x}_1)=\mathfrak{x}_2$ all pass through 0 and if either $H_1(\mathfrak{x}_1)\mathfrak{x}_1=H_2(\mathfrak{x}_2)\mathfrak{x}_2$ or $H_1(\mathfrak{x}_1)\mathfrak{x}_1=-H_2(\mathfrak{x}_2)\mathfrak{x}_2$, then $\mathfrak{x}_2=\lambda\mathfrak{x}_1$, where λ is constant and >0 in the first case, <0 in the second. (2) If the vectors $\mathfrak{x}_1$ and $\mathfrak{x}_2$ form a constant angle at 0 and $|H_1(\mathfrak{x}_1)\mathfrak{x}_1|=|H_2(\mathfrak{x}_2)\mathfrak{x}_2|$, then S_2 originates from S_1 by a similarity with center 0.

H. Busemann (Los Angeles, Calif.)

6602:

Haupt, Otto. Direkt-infinitesimalgeometrische Eigenschaften und lokale Ordnungswerte von Bogen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1958**, 1–8.

This is a collection of results dealing with curves in real Euclidean n-space. In each of these either the hypothesis satisfied by the curve is of an infinitesimal geometric nature while the implied properties concern the real order of the curve with reference to systems of order characteristics or vice versa. Where proofs are not given rather complete references are provided. Besides known results two new ones are given. The first of these is an n-dimensional generalization of a property of dually differentiable plane curves. The second gives a condition which implies that plane curves which have the same local order everywhere with respect to the parabola system $y=a_0+a_1x+\cdots+a_{k-1}x^{k-1}$ have local order k or infinity.

D. Derry (Vancouver, B.C.)

PROBABILITY

See also 6318, 6356, 6505, 6641, 6916, 7108.

6603:

Kiesow, Horst. Einige Bemerkungen zur Grundlegung der Wahrscheinlichkeitstheorie in elementaren Sprachen. Arch. Math. Logik Grundlagenforsch. **4** (1958), 124–127.

This paper contains further developments of properties of the author's simplicity principle. The principle is used in assigning a probability to a hypothesis relative both to the evidence and to the language in which hypothesis and evidence are stated. The elementary languages considered are of the same degree of complexity as those of the author's previous paper [same Arch. **4** (1958), 27–41; MR **21** #915] and as those considered by Carnap [*Logical foundations of probability*, Univ. of Chicago Press, 1950; MR **12**, 664]. *A. H. Copeland, Sr.* (Ann Arbor, Mich.)

6604:

★**Schmetterer, L. Sur l'itération stochastique.** Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 55–67. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Extensions to a Hilbert space setting of results in the theory of stochastic approximation are described. For earlier work in this direction, see J. R. Blum [Ann. Math. Statist. **25** (1954), 737–744; MR **16**, 382] and A. Dvoretzky [Proc. Third Berkeley Sympos. Math. Statist. and Probability, vol. I, pp. 39–55, Univ. of California Press, Berkeley-Los Angeles, 1956; MR **18**, 946]. Published with this paper is a discussion of it by J. Hammersley, J. Wolfowitz, E. Parzen, and P. Whittle.

D. L. Burkholder (Urbana, Ill.)

6605:

White, Robert P.; and Greville, T. N. E. On computing the probability that exactly k of n independent events will occur. Soc. Actuar. Trans. **11** (1959), no. 3, 88–99.

6606:

★**Feller, William. On combinatorial methods in fluctuation theory.** Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 75–91. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

Key results in fluctuation theory are derived by elementary combinatorial methods, e.g., the arc-sine law concerning the number of positive terms among the n first partial sums of identically distributed independent random variables, and the distribution of the maximal partial sum. The simplification of proofs of E. Sparre Andersen [Math. Scand. **1** (1953), 263–285; **2** (1954), 195–223; MR **15**, 444; **16**, 839] and F. Spitzer [Trans. Amer. Math. Soc. **82** (1956), 323–339; MR **18**, 156] is mainly due to systematic use of renewal arguments, based on ladder indices and random variables, i.e., the indices and values of the first partial sums which exceed previous partial sums. *F. L. Spitzer* (Minneapolis, Minn.)

6607:

Kiveliovitch, M. Séries chronologiques. Quelques nouveaux tests quantitatifs. J. Sci. Météorol. **10** (1958), 115–120. (English and Spanish summaries)

Let a random variable assume the values 1, 2, $\cdots$, n with given probabilities. Continuing previous work [cf. M. Kiveliovitch and J. Vialar, *Les séries chronologiques et la théorie du hasard*, Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 65, Paris, 1957; MR **19**, 897; and the references given there], the author computes the expectation of the mean of the local maxima in a series of independent observations on the random variable. Other related probabilities and expectations are also evaluated.

A. Dvoretzky (Jerusalem)

6608:

Sucheston, L. On sequences of events of which the probabilities admit a positive lower limit. J. London Math. Soc. **34** (1959), 386–394.

Let A_i, $i=1, 2, \cdots$, be a sequence of events, r a positive integer and p a positive number such that

$$\liminf P(A_{n_1}A_{n_2}\cdots A_{n_r}) \geqq p^r,$$

the lim inf understood in the sense that all indices n_j approach infinity independently. Then for any $s>r$ there exists a subsequence B_i of the A_i such that

$$\liminf P(B_{n_1}B_{n_2}\cdots B_{n_s}) \geqq p^s.$$

The bound is the best possible. The case $r=1$, $s=2$ is due to Khintchine [Compositio Math. **1** (1935), 177–179] and Visser [Bull. Amer. Math. Soc. **42** (1936), 264–286].

D. A. Darling (Ann Arbor, Mich.)

6609:

★Ullrich, Milan. Some theorems on random Schwartz distributions. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 273–291. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

The author defines a random Schwartz distribution (r.S.d.) as a generalized random variable which assumes values in the space of Schwartz distributions. For the most part the paper is concerned with questions of definition and with establishing elementary properties of r.S.d.'s. An r.S.d., as defined in this paper, is shown to be not equivalent to the definition given by K. Itô, but is shown to be a random distribution in the sense of Gelfand. Among the elementary properties proved are those concerning the derivatives of r.S.d.'s and the existence of the expected value of an r.S.d.

Stationary r.S.d.'s are studied and Itô's results on the existence and the representation of the covariance distribution of a stationary r.S.d. are shown to be valid also for the r.S.d.'s of this paper. The following strong law of large numbers is proved: If $\{\xi_n\}$ $(n=1, 2, \cdots)$ is a sequence of stationary r.S.d.'s such that (i) the expected value of ξ_1 exists, and (ii) $\{\xi_n\}$ is equicontinuous at the origin, then there exists an r.S.d. ξ_0 such that as $j\to\infty$, $j^{-1}\sum_{n=1}^{j}\xi_n(\omega)$ converges to ξ_0 with probability one.

G. Kallianpur (Bloomington, Ind.)

6610:

★Hanš, Otto. Almost sure convergence theorem for random Schwartz distributions. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 135–137. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Necessary and sufficient conditions are obtained for the almost sure convergence of a sequence of random Schwartz distributions to a random Schwartz distribution. These conditions enable the author to prove a more general version of the strong law of large numbers for stationary random Schwartz distributions than the one contained in the paper reviewed above.

G. Kallianpur (Bloomington, Ind.)

6611:

Lancaster, H. O. Zero correlation and independence. Austral. J. Statist. **1** (1959), 53–56.

Normally distributed random variables being uncorrelated does not imply that they are independent unless they are specified to have a joint normal distribution. If the components x_i of a vector x have a joint normal distribution, independence is equivalent to zero correlation for any pair of linear forms, $\sum a_ix_i$ and $\sum b_ix_i$. Zero correlation is equivalent to independence in a 2×2 contingency table. Using this fact the author proves that a necessary and sufficient condition for any two random variables x and y to be independent is that every function of x with finite variance is uncorrelated with every function of y with finite variance. Finally, if $\{x^{(i)}\}$ and $\{y^{(j)}\}$ are complete sets of orthonormal functions defined on the marginal distributions G and H of a bivariate distribution F, a necessary and sufficient condition that $F=G\cdot H$ is that $x^{(i)}$ and $y^{(j)}$ be uncorrelated for all (i, j). The first three statements above have been generally known among statisticians for some time.

H. Chernoff (Stanford, Calif.)

6612:

Trybula, S. On lower bounds for probability moments. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 267–269. (Russian summary, unbound insert)

Suppose that the random variable x has an absolutely continuous distribution with density function $f(x)$. The nth probability moment π_n is defined by $\pi_n=\int_{-\infty}^{\infty}f^n(x)dx$. The author determines the greatest lower bound of π_n for random variables with finite kth central absolute moment m_k $(n>1, k\geqq 1)$.

L. A. Aroian (Los Angeles, Calif.)

6613:

Tucker, Howard G. A generalization of the Glivenko-Cantelli theorem. Ann. Math. Statist. **30** (1959), 828–830.

The Glivenko-Cantelli Theorem states that if $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent, identically distributed random variables with an arbitrary common distribution function $F(x)$, then the sequence $\{F_n(x)\}$ of empirical distribution functions converges uniformly to $F(x)$ with probability one. [See Loève, *Probability theory*; Van Nostrand, New York, 1955; MR **16**, 598; and Gnedenko, *Kurs teorii veroyatnostei*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad,

1950; MR **13**, 565.] As the author points out, independence is an unnecessary assumption, and the conclusion holds if the sequence of random variables is a strictly stationary, ergodic (metrically transitive) one. Using the ergodic theorem for random variables [see Loève, op. cit., p. 434], the author proves a generalization of the Glivenko-Cantelli Theorem when the sequence of random variables is strictly stationary, not necessarily ergodic, and the common distribution function is arbitrary.

H. P. Edmundson (Pacific Palisades, Calif.)

6614:

Böge, Werner. Über die Charakterisierung unendlich teilbarer Wahrscheinlichkeitsverteilungen. J. Reine Angew. Math. **201** (1959), 150–156.

Es sei G eine beliebige endliche multiplikativ geschriebene Gruppe und $R(G)$ der Gruppenring über dem Körper der reellen Zahlen. Es sei Z die Gesamtheit der Elemente aus $R(G)$, deren Koeffizienten nicht negativ sind und deren Koeffizientensumme 1 ist. Die Teilmenge B von Z sei so definiert: $b \in Z$ gehört genau dann zu B, wenn es zu jedem natürlichen k ein $a_k \in Z$ gibt mit $b = a_k^k$. B kann man als Gesamtheit der unendlich oft teilbaren Wahrscheinlichkeitsverteilungen über G auffassen. Sei H eine Untergruppe von G und R_H die Gesamtheit der bezüglich H zweiseitig invarianten Elemente von $R(G)$. Führt man die übliche Ringnorm in $R(G)$ ein, wird $R(G)$ eine Banach-Algebra. Für jede Untergruppe H definiere man nun in üblicher Weise die Exponentialfunktion $\exp_H$ über R_H. Einer Anregung des Referenten folgend, zeigt der Verf.: Es sei K_H die Menge der $v = \sum_{g \in G} v_g g$ in R_H mit $v_g \geq 0$ für $g \notin H$ und $\sum_{g \in G} v_g = 0$. Es sei B_H die Menge aller Elemente in $R(G)$ der Gestalt $\exp_H v$ mit $v \in K_H$. Dann ist $B = \bigcup_H B_H$. B_H läßt sich übrigens auch kennzeichnen als Durchschnitt von B mit der Gruppe der regulären Elemente von R_H. Weiter wird das Problem der Eineindeutigkeit der Exponentialdarstellung studiert. Dadurch wird einerseits der Satz erhalten, daß B genau dann multiplikativ abgeschlossen ist, wenn G kommutativ ist und andererseits erfahren frühere Eindeutigkeitsaussagen von Vorob'ev [Mat. Sb. (N.S.) **34** (**76**) (1954), 89–126; MR **15**, 882] für abelsche Gruppen eine Korrektur. Es sei noch auf kürzlich veröffentlichte Untersuchungen von Urbanik [Colloq. Math. **6** (1958), 13–24; MR **21** #2919] hingewiesen.

L. Schmetterer (Hamburg)

6615:

Derman, C.; and Sacks, J. On Dvoretzky's stochastic approximation theorem. Ann. Math. Statist. **30** (1959), 601–606.

The theorem in question was proved by the reviewer [Proc. Third Berkeley Sympos. on Math. Statist. and Prob. 1954–55, vol. I, pp. 39–55, Univ. of California Press, Berkeley-Los Angeles, 1956; MR **18**, 946] and another, somewhat similar proof was given by J. Wolfowitz [Ann. Math. Statist. **27** (1956), 1151–1155; MR **19**, 185]. The theorem asserts convergence both in mean square and with probability 1. Here the authors give a considerably simpler and shorter proof of the probability 1 assertion. The key to the new proof is the following easily established lemma: Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\xi_n\}$ be sequences of non-negative numbers and $\{\delta_n\}$ be a sequence of real numbers satisfying $a_n \to 0$, $\sum b_n < \infty$, $\sum c_n = \infty$, $\sum \delta_n$ convergent and $\xi_{n+1} \leq \max(a_n, (1+b_n)\xi_n + \delta_n - c_n)$; then $\xi_n \to 0$. A similar lemma is used to give an explicit extension of the (probability 1 part of the) approximation theorem to the multi-dimensional case.

A. Dvoretzky (Jerusalem)

6616:

★Régnier, André. Sur les perturbations aléatoires des systèmes d'équations différentielles. Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 149–155. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

The author considers stochastic perturbations of the system of differential equations $dx_t/dt = u(x_t, a)$, where x_t and a are vectors, when a varies stochastically. Under suitable assumptions conditions are obtained for the ensuing x_t process to be of a certain kind, e.g., stationary.

A. Dvoretzky (Jerusalem)

6617:

Mercer, A. Some simple duration-dependent stochastic processes. J. Roy. Statist. Soc. Ser. B **21** (1959), 144–152.

Let $X(t)$, $t \geq 0$, be a stochastic process assuming the two values 1 and 2, with $\Pr\{X(0) = 1\} = k$, and let $T(t)$ be the length of time $X(\tau)$ has spent in the state 1 during $0 \leq \tau \leq t$. The author assumes the vector process $(X(t), T(t))$ is jointly Markovian, and determined by a pair of functions $f_i(t)$, $i = 1, 2$, in which the probability for a change from the state i during $(\tau, \tau + d\tau)$, given that $X(\tau) = i$ and $T(\tau) = t$, is $f_i(t)d\tau + o(d\tau)$. By the usual methods differential equations are deduced which are satisfied by

$$p_i(t, \tau) = \frac{d}{dt} \Pr\{X(\tau) = i,\ T(\tau) < t\}$$

and its Laplace transforms on τ. Only the case $f_i(t) = \alpha_i t^{\beta_i}$ where the β_i are non-negative integers is treated, and certain limits ($t \to \infty$) are examined for this case.

D. A. Darling (Ann Arbor, Mich.)

6618:

Kellerhals, Gerhard. Randwertprobleme bedingter Wahrscheinlichkeiten. Math. Z. **71** (1959), 1–30.

A family of events indexed by $t \in T$ where T is a linearly or cyclically ordered set has the nearest neighbor property if the conditional probability of any one of them, given some preceding and some following events, reduces to the conditional probability given the nearest preceding and the nearest following ones [see, for example, E. Montroll, J. Chem. Phys. **9** (1941), 706–721; **10** (1942), 61–77]. It is well known [see, for example, Bartlett, *An introduction to stochastic processes*, University Press, Cambridge, 1955; MR **16**, 939; p. 35] that when T is linearly ordered the above property reduces to the Markov property. The author proceeds to elementary manipulations of the nearest neighbor property when T is a finite set and when T is a circumference: To recursion formulas he obtains in the first case, correspond in the second case functional equations and Riccati-type differential equations.

M. Loève (Berkeley, Calif.)

6619:

Whittle, P. Continuous generalizations of Tchebichev's inequality. Teor. Veroyatnost. i Primenen. **3** (1958), 386–394. (Russian summary)

Using recent generalizations of Chebyshev's inequality to multivariate distributions [Olkin and Pratt, Ann. Math. Statist. **29** (1958), 226–234; MR **20** #385; Whittle, Quart. J. Math. Oxford Ser. (2) **9** (1958), 232–240; MR **20** #6754], the author considers the problem of obtaining a lower bound for $P = \text{Prob}\{|X(t)| \leq \alpha(t);\ 0 \leq t \leq 1\}$, where $X(t)$ is a stochastic process with given expectation and covariance function and $\alpha(t)$ a given positive function. He obtains a number of results, the most specific being this: assume without loss of generality $E[X(t)] = 0$, $\alpha(t) = \alpha = \text{constant}$, and denote $E[X(s), X(t)] = v(s, t)$, $\varphi^2(t) = v(t, t)$, $\psi^2(t) = [\partial^2 v(s, t)/\partial s \partial t]_{s=t}$; then under certain regularity conditions

$$P \leq \frac{\varphi^2(0)+\varphi^2(1)}{2\alpha^2}+\frac{1}{\alpha^2}\left[\int_0^1 \varphi^2(t)dt \int_0^1 \psi^2(t)dt\right]^{1/2}.$$

Z. W. Birnbaum (Seattle, Wash.)

6620:

★Kac, M. Some remarks on stable processes with independent increments. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 130–138. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

Let $x(t)$ $(x(0) = 0)$ be the process with stationary independent increments governed by the symmetric stable density $\rho_\alpha(x)$ of index $1 \leq \alpha \leq 2$. The author establishes relations between the integral operator (difference kernel) $\rho_\alpha(x-y)$, $-T \leq x, y \leq T$, and the probabilities

$$\text{Prob}\,[-A \leq x+x(\tau) \leq A,\ 0 \leq \tau \leq t],$$

which suggest conjectures concerning the asymptotic behavior of the eigenvalues $\lambda_l(T)$, $T \to \infty$, of the above integral operator. On the other hand, the probabilistic absorption problem defines a contraction semi-group which is known to have an eigenfunction expansion with eigenvalues ν_l. A partly heuristic argument suggests that $\nu_l = \mu_l$, where $\lambda_l(T) = 1 - \mu_l T^{-\alpha} + O(T^{-\alpha})$. This is known when $\alpha = 2$.

Meanwhile a related conjecture, concerning the behavior of ν_j, as $j \to \infty$, has been proved by R. M. Blumenthal and R. K. Getoor [Pacific J. Math. **9** (1959), 399–408; MR **21** #6023]. Another conjecture to the effect that $T_x = C_\alpha(1-x^2)^{\alpha/2}$ when T_x is the expected time until the process $x+x(t)$ leaves $(-1, 1)$, has been proved by R. K. Getoor (unpublished) with

$$C_\alpha = \pi^{1/2}\left[2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)\right]^{-1}.$$

F. L. Spitzer (Minneapolis, Minn.)

6621:

Legoupil, Jean. Sur la dépendance de probabilité entre une variable aléatoire et une fonction aléatoire. C. R. Acad. Sci. Paris **249** (1959), 1444–1446.

An announcement of 6 theorems concerning the regression of a random variable Y on a stochastic process $X(t)$, $a \leq t \leq b$. Let Y and $X(t)$ have second order moments, and let B be the greatest lower bound of

$$B_\rho = E\{(Y - \int_a^b \rho(t)X(t)dt)^2\}, \quad \rho \in L_2(a, b).$$

The main theorem gives conditions under which B can be attained and under which the corresponding ρ_0 is unique. If the conditions are not met there is at least one sequence ρ_n such that $B_{\rho_n} \to B$ and $\int_a^b \rho_n(t)X(t)dt$ converges in the mean square. *D. A. Darling* (Ann Arbor, Mich.)

6622:

★Cramér, Harald. Remarques sur le problème de prédiction pour certaines classes de processus stochastiques. Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 103–112. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Let x_n $(n = 0, \pm 1, \pm 2, \cdots)$ be a second-order stochastic process with mean values zero and covariances $Ex_m\bar{x}_n = r(m, n)$. There exists an increasing sequence of integers j_k and orthonormal variables ξ_{j_k} such that

$$x_n = u_n + v_n \quad (u_n = \sum_{j_k \leq n} C_{nj_k}\xi_{j_k}).$$

Here v_n is a deterministic process, $v_n \perp u_m$, the C's are complex constants and the sum converges in mean square. The best linear predictor x^*_{n+p} of x_{n+p} in terms of x_n, $x_{n-1}, \cdots$ is given by

$$x^*_{n+p} = \sum_{j_k \leq n} C_{n+p,j_k}\xi_{j_k} + v_{n+p}$$

with the prediction error $\|x^*_{n+p} - x_n\|^2 = \sum_{n < j_k \leq n+p} |C_{n+p,j_k}|^2$. This generalizes the well-known decomposition due to Wold.

Let x_n be a harmonizable process with $r(m, n) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(mu-nv)}dF(u, v)$, where F is of bounded variation on the region of integration. Form the one-dimensional measure $dG(u)$ by projecting the measure $|dF(u, v)|$ on the u-axis. Then $\int_{-\pi}^{\pi} \log|G'(u)|du = -\infty$ is a sufficient condition for the x_n-process to be deterministic.

Finally a condition is given ensuring that a sequence j_k should belong, in the manner described above, to a given harmonizable process.

The author has informed the reviewer that the inequality for the prediction errors σ_{np} given on p. 105 should read $0 \leq \sigma_{n-1,1} \leq \sigma_{n-2,2} \leq \cdots$. In order that the inequality as given in the paper should hold, the definition of σ_{np} should be appropriately modified.

U. Grenander (Stockholm)

6623:

★Neveu, Jacques. Sur le comportement asymptotique des chaînes de Markov. Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 185–193. Colloque Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Using the method of Riesz spaces the author treats certain lattice properties of "invariant" quantities in Markov chains. These quantities are related to the "almost closed" sets considered by Blackwell [Ann. Math. Statist. **26** (1955), 654–658; MR **17**, 754] in the case of stationary transition and are of interest mainly for non-recurrent (transient) behavior of the chain.

K. L. Chung (Syracuse, N.Y.)

6624:

Kozin, Frank. On the quadratic detector. I. J. Math. Phys. **38** (1959/60), 119–125.

The author treats the problem of transforming the distribution of $\int_0^T K(T-t)x^2(t)dt$, where $x(t)$ is an infinitely divisible process, to the distribution of a similar functional involving the Wiener process.

D. A. Darling (Ann Arbor, Mich.)

6625:

Neveu, Jacques. Un exemple de processus markoviens stationnaires. C. R. Acad. Sci. Paris **249** (1959), 1309–1311.

The author gives a method for constructing a class of stationary Markov chains with denumerable state space E and transition probabilities $P_\tau(i,j)$ having the property

$$(*) \qquad \lim_{\tau\to 0}\frac{d}{d\tau}P_\tau(i,j) = q_i\delta_{ij} \quad (0 < q_i < \infty).$$

It is asserted that the class prescribed is the most general process satisfying (*) and having the property that the set of positive bounded measurable solutions of

$$f_{\sigma+\tau}(j) = \sum_{i\in E} f_\sigma(i)P_\tau(i,j) \quad (j\in E;\ \sigma,\tau > 0)$$

with $\lim_{\tau\to 0} f_\tau(j) = 0$ $(j\in E)$ forms a cone of finite dimension.

J. L. Snell (Hanover, N.H.)

6626:

Watanabe, Takesi. Some general properties of Markov processes. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 9–29.

The author considers time homogeneous Markov processes with state space E a separable locally compact space. Conditions are obtained that such a process be conservative, i.e., $P(t,x,E)=1$, for $t\geq 0$. These conditions are expressed in terms of sojourn sets introduced by Feller [Trans. Amer. Math. Soc. **83** (1956), 19–54; MR **19**, 892] and the Green measure $G(x,\cdot)=\int_0^\infty P(t,x,\cdot)dt$. As a special case the author devises the conditions given by Feller [ibid. **48** (1940), 488–515; MR **2**, 101] for a Markov chain to be conservative. The author also gives conditions in terms of the Green measure that a process be recurrent at a point x. The basic condition is that the Green measure of every open set containing x should be infinite.

J. L. Snell (Hanover, N.H.)

6627:

★Lévy, Paul. Construction du processus de W. Feller et H. P. McKean en partant du mouvement Brownien. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 162–174. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

Proofs, rectifications, and further discussion of the results in a previous paper [C. R. Acad. Sci. Paris **245** (1957), 1772–1774; MR **19**, 1202].

J. L. Doob (Urbana, Ill.)

6628:

Wolff, Karl-H. Die Ruinwahrscheinlichkeit in der kollektiven Risikotheorie für spezielle Schadensverteilungen. Statist. Vierteljschr. **10** (1957), 163–176.

Es sei für $y\geq 0$ $w(y)$ die Ruinwahrscheinlichkeit, d.h. die Wahrscheinlichkeit, daß unter den üblichen Voraussetzungen der kollektiven Risikotheorie eine Gesellschaft, der das Anfangskapital y zur Verfügung steht, irgendwann im Laufe der Zeit zahlungsunfähig wird. Die Schadensverteilungsfunktion F soll vom Zeitpunkt des Schadenseintritts unabhängig und $F(0)=0$ sein. $a(s)\int_0^\infty e^{-sv}dw(y)$ konvergiere in einer Halbebene $R(s)>g$ mit $g<0$. Der Autor nimmt an, daß F eine Dichte f besitzt und untersucht die Gestalt von w und von f, wenn sich $a(s)$ zu einer rationalen Funktion in der komplexen s-Ebene fortsetzen läßt. Das Studium solcher Sonderfälle hat praktisches Interesse.

L. Schmetterer (Hamburg)

6629:

Austin, D. G. The generalized backward Kolmogorov equation in abstract space. Illinois J. Math. **3** (1959), 532–537.

Let $(\mathcal{F}, x)$ denote an abstract space X and a Borel field $\mathcal{F}$ of subsets of X containing X and all one-point sets, and let $P_t(x,E)$ $(t\geq 0, x\in X, E\in\mathcal{F})$ denote a stationary Markov transition function such that $P_t(x,\{x\})$ is $\mathcal{F}$-measurable for each t, tends to $P_0(x,\{x\})$ $(=1)$ as $t\downarrow 0$ for all x, and has a finite right-hand derivative at $t=0$ for some one x (x^*, say). Then the derivative $P_t'(x^*,E)$ exists for $t>0$ and all E in $\mathcal{F}$, and $P_t'(x^*,\cdot)$ is a uniformly bounded signed measure on $\mathcal{F}$ satisfying

$$P'_{t+s}(x^*,E) = \int_X P_s(\cdot,E)dP_t'(x^*,\cdot).$$

{The very intricate argument contains a number of slips and misprints.}

D. G. Kendall (Oxford)

6630:

Finch, P. D. Cyclic queues with feedback. J. Roy. Statist. Soc. Ser. B **21** (1959), 153–157.

The author considers two types of cyclic queues with feedback. In both cases, customers arrive with a Poisson arrival rate and must be served by each of m servers (with mean service rates μ_j, $j=1,\cdots,m$, exponential distribution) before they can depart from the system. In the first case, there is a probability p_j, $j=1,\cdots,m$, that after being served by the mth server a customer will have to return to the jth server and start service from there on again. In the second case the probability p_j represents the probability that a customer after completing service at the jth server will have to receive service at that server again. The author is only concerned with the number of customers in front of each server. He obtains, by setting up the standard set of equations, steady state solutions for $P(n_1,n_2,\cdots,n_m)$ under the constraint that there can be no more than N customers in the system at any one time (n_j is the number of customers in front of the jth server).

The solution for the first case is:

$$(1) \quad P(n_1,n_2,\cdots,n_m) = (x_1)^{n_1}(x_2)^{n_2}\cdots(x_m)^{n_m}P(0,\cdots,0),$$

where $x_j=\lambda(1-p+p_1+\cdots+p_j)/(\mu_j(1-p))$, $j=1,\cdots,m$, and $p=\sum_{j=1}^m p_j$.

The solution for the second case has the same form as (1) with $x_j=\lambda/(\mu_j q_j)$, where $q_j=1-p_j$.

In both cases $P(0,\cdots,0)$ is found by normalization, and explicit values are found for the average number of

customers and the average number of waiters at each stage for the limiting case, $N = \infty$.

The author also treats the single server case with general service time (here both types mentioned above are the same). *H. M. Gurk* (Princeton, N.J.)

6631:

Winsten, C. B. Geometric distributions in the theory of queues. J. Roy. Statist. Soc. Ser. B **21** (1959), 1–35.

The author considers some queueing problems in which the queue length forms a Markov chain with a transition matrix which, except for the first few columns, is a Toeplitz matrix and in which the queue can increase at any stage by at most one unit. The equilibrium distribution (if it exists) is shown to be a geometric distribution except possibly for the first few states. The proof of this is obtained by an interesting probabilistic argument about time paths of the queue rather than by the more conventional use of the symmetry of the matrix. The paper treats in some detail a process in which service has an exponential distribution and customers arrive at integer time points or at integer times plus a random time less than either one or two. The latter cases represent a queueing model for customers who arrive late for scheduled appointments. The paper is followed by several discussions including some generalizations and mathematical simplifications by L. Takács. *G. Newell* (Providence, R.I.)

6632:

Heathcote, C. R. The time-dependent problem for a queue with preemptive priorities. Operations Res. **7** (1959), 670–680.

The author discusses a queueing system of the general type $M/M/1$, but in which there are two Poisson streams of incoming customers, members of one of which have overriding priority, so that the service of a non-priority customer is liable to interruption by the arrival of a member of the priority class. The steady-state behaviour of this system was discussed by H. White and L. S. Christie [Operations Res. **6** (1958), 79–96; MR **19**, 1091] and by F. F. Stephan [ibid. **6** (1958), 399–418; MR **19**, 1203]. Here the author determines the joint distribution at a general time t of the numbers of priority and non-priority customers in the system, for given initial conditions. (Detailed formulas are worked out only in the special case of equal service rates for the two classes—the algebra then simplifies.) Busy-period phenomena are then studied for the queue of non-priority customers.

D. G. Kendall (Oxford)

6633:

Gani, J.; and Prabhu, N. U. Remarks on the dam with Poisson type inputs. Austral. J. Appl. Sci. **10** (1959), 113–122.

The initial models for dams introduced by Moran [same J. **5** (1954), 116–124; MR **16**, 269] for discrete time and later extended by the authors to continuous time are essentially Markov processes and closely connected with Takàcs' queueing processes, whose method helps to simplify the respective investigations.

The present paper deals with the continuous time equations resulting from applying these methods to a dam model of capacity K with simple Poisson inputs, a steady release and an overflowing occurring whenever the contents exceed the capacity.

Dams with inputs having an infinitely divisible distribution law of Poisson type are also considered.

O. Onicescu (Bucharest)

6634:

Wishart, David M. G. A queuing system with service-time distribution of mixed chi-squared type. Operations Res. **7** (1959), 174–179.

G. Luchak [Operations Res. **4** (1956), 711–732; MR **18**, 707] has discussed the queueing system $M/L/1$ (a Poissonian input, a single server, and a service-mechanism such that some constant multiple of the service-time v is distributed according to a mixture of chi-square distributions, each component having an even integral number of degrees of freedom), and has noted the following interpretation; if the service-mechanism is such that μv has the distribution of

$$\tfrac{1}{2}\sum_{m=1}^{k} c_m \chi^2_m{}^2 \quad (c_m \geqq 0, \sum c_m = 1),$$

then we can describe this state of affairs by saying that an arriving customer has probability c_m of demanding m phases of service, the time for completion of each phase having the distribution $e^{-x}\mu dx$ $(0 < x < \infty)$.

The author here considers the more general system $GI/L/1$, and shows that if attention is directed to the number of phases of service awaiting completion and taking priority over a customer at the epoch of his arrival, then the method of the imbedded Markov chain can be employed, and in fact the calculations are very similar to those already carried out by him in his study of the system $GI/E_k/1$. He is therefore able to adapt his earlier work [Ann. Math. Statist. **27** (1956), 768–779; MR **18**, 157] to the present problem and to calculate the equilibrium waiting-time distribution, etc., for $GI/L/1$. As an application he studies the system $D/E_{2\cdot5}/1$ studied by Bailey [J. Roy. Statist. Soc. Ser. B **14** (1952), 185–199]; the service-time distribution $E_{2\cdot5}$ cannot be handled by the usual method, but the author shows that it can be approximated by an L-type distribution.

D. G. Kendall (Oxford)

6635:

Gaver, Donald P., Jr. Imbedded Markov chain analysis of a waiting-line process in continuous time. Ann. Math. Statist. **30** (1959), 698–720.

The author discusses the transient and the limiting behaviour of a generalisation of the queueing system $M/G/1$ in which the arrivals form bunches of random size; in fact the input is a stationary compound Poisson process. In terms of multiple generating functions, Laplace transforms and Laplace-Stieltjes transforms he obtains formulae determining (i) the joint distribution of busy period duration and number of departures therein, (ii) the n-step "zero-avoiding" transition probabilities, and (iii) the joint d.f. of the number of customers present in the system at a fixed time t after the beginning of a busy period and the number who have been serviced by that time. The effect of "instantaneous defections" is also discussed. The author then discusses the process as a whole by regarding the time-axis as made up of a succession of independent idle and busy periods, making use of his earlier results about the behaviour of the process

during a busy period, and finally, following W. L. Smith [Proc. Cambridge Philos. Soc. **51** (1955), 629–638; Proc. Royal Soc. Edinburgh Ser. A **64** (1954), 9–48; MR **17**, 165; **15**, 722], he applies renewal theory to study the ergodic behaviour of his system, and investigates the asymptotic forms of the equilibrium distributions for queue-size and waiting time. *D. G. Kendall* (Oxford)

6636:

Kac, M. Remark on recurrence times. Phys. Rev. (2) **115** (1959), 1.

"A simple derivation is given of the mean recurrence time of a linear chain of harmonically bound particles. The derivation is based on a formula of Smoluchowski." [Author's summary.] The author's remark was prompted by a paper by P. Chr. Hemmer, L. C. Maximon, and H. Wergeland [same Rev. **111** (1958), 689–694; MR **20** #1443] in which the stated result was derived differently. There appears to be a slip in the present author's argument; it is thought that the right-hand side of the third displayed formula should read

$$\tau \sum_{j=1}^{N-1} \frac{w_j}{\Delta\varphi_j} \prod_{k=1}^{N-1} \frac{\Delta\varphi_k}{2\pi}.$$

D. G. Kendall (Oxford)

6637:

Levinson, Norman. Limiting theorems for Galton-Watson branching process. Illinois J. Math. **3** (1959), 554–565.

The author considers the Galton-Watson branching process with probability p_j that one object is replaced by j objects in the next generation, $f(s)=\sum_{j=0}^{\infty} p_j s^j$, $f'(1)=m$. The case $1<m<\infty$ is considered. Exclude the trivial case $f(s)=s^m$. Let z_n be the number in the nth generation, with $z_0=1$. The generating function of z_n is f_n, the nth iterate of f. Let $w_n=z_n/m^n$, $\phi_n(s)=E(e^{-sw_n})$, $\mathrm{Re}\, s\geq 0$. If $Ez_1^2<\infty$, it is known that w_n converges in distribution and in the mean (order 2) to a random variable w, $\phi_n(s)\to\phi(s)$ where $\phi(s)=Ee^{-sw}$, w has an absolutely continuous distribution except possibly at 0, and $Ew=1$ [Harris, Ann. Math. Statist. **19** (1948), 474–494; MR **10**, 311]. J. L. Doob pointed out that since the w_n are a martingale and Ew_n is bounded, w_n converges to a random variable w with probability 1. However, if only the first moment Ez_1 is finite, then it is not clear whether $Ew=1$. The present paper gives an example where $Ew=0$, and gives a condition weaker than the existence of a second moment which implies $Ew=1$. The condition is that there exists a constant C and a function $\gamma(t)$ that is continuous and monotone nondecreasing for $t\geq 1$ with $\int_1^\infty (1/t\gamma(t))dt<\infty$, such that $\sum_{j=n}^{\infty} jp_j\leq C/\gamma(n)$. The basic step in the proof is showing that this condition implies the existence of a continuous nondecreasing function $\alpha(\sigma)$ $(0\leq\sigma<\infty,\ \alpha(0)=0)$ such that $\int_0^1 (\alpha(\sigma)/\sigma)d\sigma<\infty$ and $\left|(1-f(e^{-\sigma}))/\sigma-m\right|\leq\alpha(\sigma)$ $(0\leq\sigma<\infty)$. From this it is shown, from the relation $\phi_{n+1}(s)=f(\phi_n(s/m))$, that on any finite interval of $0\leq s<\infty$, $\phi_n(s)$ converges uniformly to a limit $\phi(s)$ with $\phi(0)=1$ and $\lim_{s\to 0+}(1-\phi(s))/s=1$. Moreover, ϕ is uniquely determined by the functional equation $\phi(s)=f[\phi(s/m)]$ and the conditions $|\phi|\leq 1$, $\phi(0)=1$, $\phi'(0)=-1$, even if s is restricted to $s\geq 0$. It is also shown that the distribution $G(u)=P(w\leq u)$ has a continuous derivative $G'(u)$ except possibly at $u=0$, and that $G(0+)$ is the probability that $z_n=0$ for some n. *T. E. Harris* (Santa Monica, Calif.)

6638:

Caughey, T. K. Response of Van der Pol's oscillator to random excitation. J. Appl. Mech. **26** (1959), 345–348.

The power spectrum of the noise output of a Van der Pol oscillator is determined under the assumption that the noise amplitude is much smaller than the signal amplitude. Implicit is the assumption that the tank circuit has a narrow bandwidth so that noise components at frequencies far from the resonance frequency can be disregarded. A linear equation is obtained for the noise excitation, with a coefficient dependent upon the signal amplitude. It is found that for a fixed applied noise current (a) the mean squared noise-voltage amplitude is inversely proportional to the signal amplitude; (b) the noise bandwidth increases with increasing amplitude of self-oscillation. The results are checked by an analog-computer solution of the fundamental equation. *H. A. Haus* (Wien)

6639:

Litwiniszyn, Jerzy. Flows in pipe networks from the point of view of the theory of random processes. Arch. Mech. Stos. **11** (1959), 421–440. (Polish and Russian summaries)

Guided by the concrete problem of the flow of smoke in mine ventilation systems, the author constructs a general theory of random flow in pipe networks. The mathematical ingredients of the article are network topology and Markov process theory. As applied to the above-mentioned concrete problem, the theory allows one to calculate the probability of appearance of smoke at any node of a pipe network at time t, given the smoke distribution among the nodes of the network at any earlier time. *R. A. Silverman* (New York, N.Y.)

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See also 6638, 6740, 6935, 7102, 7123, 7126, 7127.

6640:

Talacko, Joseph. A note about a family of Perks' distribution. Sankhyā **20** (1958), 323–328.

The author classifies and discusses distributions with c.d.f. of the form $h_2(x)=A_1/(b_0e^x+k+b_0e^{-x})$. *R. F. Tate* (Seattle, Wash.)

6641:

Agnew, Ralph Palmer. Asymptotic expansions in global central limit theorems. Ann. Math. Statist. **30** (1959), 721–737.

Consider the distribution function (d.f.) $F_n(x)$ of the normalized sum

$$(\xi_1+\xi_2+\cdots+\xi_n)/n^{1/2},$$

where the ξ's are independent random variables with the same d.f., $F(x)$, having zero mean and unit variance. This paper studies the asymptotic behavior of constants $C_n(p)$, where

$$C_n(p)=\int_{-\infty}^{\infty} |F_n(x)-\Phi(x)|^p dx,$$

$\Phi(x)$ being the Gaussian d.f. Primary interest is focused on the case $p=2$, for which conditions are presented under which there exist constants $\{D_i\}$ such that

$$C_n(2) = C_n = \sum_{i=1}^{k} D_i n^{-i} + O(n^{-k-1}).$$

Different approaches are required depending on whether or not the magnitude of the characteristic function (c.f.) of $F(x)$ is periodic. It is shown that $D_i = A_i + B_i$, where B_i is needed only when the c.f. is periodic. Formulas are given for A_1 and A_2 and for B_1 and B_2, when required, in terms of the first five cumulants of $F(x)$. Methods extending the results to $k>2$ are indicated.

Specific results are presented for a uniform d.f. (non-periodic) and a binomial d.f. (periodic). Extensive computations are made for the symmetric binomial d.f., with results for $k \leq 5$.

Values of C_n are presented for $n \leq 10$ and a very good approximation is available for $n \geq 5$.

{Note the transposition of digits for D_1 in (7.61).}

R. L. Anderson (Raleigh, N.C.)

6642:

Kiefer, J.; and Wolfowitz, J. Asymptotic minimax character of the sample distribution function for vector chance variables. Ann. Math. Statist. **30** (1959), 463–489.

The authors, in a joint paper with the reviewer [Dvoretzky, Kiefer and Wolfowitz, same Ann. **27** (1956), 642–669; MR **18**, 772] have shown that for a very general class of weight functions the sample distribution is an asymptotically minimax estimator of the distribution of a real valued random variable. In the present paper similar results are obtained for (finite dimensional) vector-valued random variables. The method of the quoted paper must be greatly modified since the "distribution-free" properties of the sample d.f. no longer hold. The authors utilize their previous results [Trans. Amer. Math. Soc. **87** (1958), 173–186; MR **20** #5519] on the deviation of the sample d.f. from the true d.f. in the multi-dimensional case. From the uniformity of approach of the d.f. of the normalized maximum deviation to its limit the authors deduce the existence of an approximately least favorable distribution and proceed hence to prove the following result: Let $W(r)$, for $r \geq 0$, be continuous, non-negative, non-decreasing, not identically zero and satisfy (*) $\int_0^\infty rW(r)e^{-c_m' r^2}dr < \infty$ (c_m' being a positive constant depending only on m); then the sequence of sample d.f. is asymptotically minimax relative to the sequence of weight functions $W_n(F, g) = W(n^{1/2} \sup_z |g(z) - F(z)|)$ and the family of all m-dimensional d.f. F. A similar result, with c_m' replaced by 2, is proved for integral weights $W_n(f, g) = \int W(n^{1/2})F(x) - g(x)|)dx$. Generalizations to other loss functions, etc., are indicated.

A. Dvoretzky (Jerusalem)

6643:

Ziaud-Din, M. The expression of k-statistic k_{11} in terms of power sums and sample moments. Ann. Math. Statist. **30** (1959), 825–828.

The eleventh k-statistic is given in terms of the power sums of the observations. The formula is two and a half pages long. *S. W. Nash* (Vancouver, B.C.)

6644:

Federighi, Enrico T. Extended tables of the percentage points of Student's t-distribution. J. Amer. Statist. Assoc. **54** (1959), 683–688.

The author states that in using Student's t-distribution in testing component parts a need for extending the table of upper percentage points was revealed. The method of calculation of these percentage points is presented and a table containing these results is given. Let y_t be the elementary density for Student's t with n degrees of freedom, and denote $\int_{t_0}^\infty y_t dt = P$. The values of t_0 are given to three decimal places for $P = .25, .10, .05, .025, .01, .005, .0025, .001, 5\times10^{-4}, 25\times10^{-5}, 1\times10^{-4}, 5\times10^{-5}, 25\times10^{-6}, 1\times10^{-5}, 5\times10^{-6}, 25\times10^{-7}, 1\times10^{-6}, 25\times10^{-8}, 1\times10^{-7}$, and $n = 1(1)\ 30(5)\ 60(10)\ 100,\ 200,\ 500,\ 10^3,\ 2\times10^3, 10^4$. It would have been advantageous had the large values of n been arranged conveniently for harmonic interpolation, such as $n = 60, 120, 240, 480, 960$, etc.

L. A. Aroian (Los Angeles, Calif.)

6645:

Jílek, Miloš; and Líkař, Otakar. Coefficients for the determination of one-sided tolerance limits of normal distribution. Ann. Inst. Statist. Math. Tokyo **11** (1959), 45–48.

A table of coefficients k for the determination of one-sided tolerance limits of normal distribution have been calculated for $n = 5(1)\ 20(5)\ 50(10)\ 100(100)\ 300, 500, 1000$ for $P = 0.90, 0.95, 0.99$ and $\alpha = 0.90, 0.95, 0.99$ with the help of non-central t-distribution table.

J. Janko (Prague)

6646:

Ishii, Goro. On the exact probabilities of Renyi's tests. Ann. Inst. Statist. Math. Tokyo **11** (1959), 17–24.

Let X be a real valued random variable with continuous d.f. $F(x)$ and $X_1 \cdots X_n$ be n independent "observations" on X. $F_n(x)$ is the empirical d.f. of this sample, i.e.,

$$F_n(x) = \frac{\text{Number of } X_i \leq x}{n}.$$

Renyi [Acta. Math. Acad. Sci. Hungar. **4** (1953), 191–231; MR **15**, 885] has given the limiting distribution of statistics of the form

$$\sup_{a \leq F(x)} \frac{F_n(x) - F(x)}{F(x)}.$$

By extending a lemma due to Birnbaum and Tingey [Ann. Math. Statist. **22** (1951), 592–596; MR **13**, 367], the distribution of this and related statistics are determined for finite n. The results are expressed as double sums.

D. G. Chapman (Seattle, Wash.)

6647:

Graybill, Franklin A.; and Deal, R. B. Combining unbiased estimators. Biometrics **15** (1959), 543–550.

The problem considered is that of combining linearly two independent unbiased estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$ of a parameter α when the ratio of the variances σ_1^2, σ_2^2 of $\hat{\alpha}_1$, $\hat{\alpha}_2$ is unknown. The combined estimator $\hat{\alpha}$ is termed by the authors a uniformly better unbiased estimator of α (than $\hat{\alpha}_1$, $\hat{\alpha}_2$) if its variance is less than or equal to the smaller of σ_1^2, σ_2^2 for all values of σ_1^2, σ_2^2. It is well known that no linear combination with constant weights can be uniformly

better but under certain circumstances it is possible to find random weights which achieve this. An important special case is studied and the following main result is obtained. Let x, y be normal variates with common mean μ and unknown variances σ_1^2/n_1, σ_2^2/n_2; let $m_1 s_1^2/\sigma_1^2$, $m_2 s_2^2/\sigma_2^2$ be distributed as χ^2 with m_1, m_2 degrees of freedom and suppose all random variables are independent. Then a necessary and sufficient condition that

$$\hat{\mu} = (n_1 s_2^2 x + n_2 s_1^2 y)/(n_1 s_2^2 + n_2 s_1^2)$$

is an unbiased estimator of μ which is uniformly better than either x or y is that m_1 and m_2 are both larger than 9.

An application of the theorem is made to the recovery of interblock information in a balanced incomplete block design. This information should be used if (i) $b-t=9$, $rt-b-t+1 \geq 18$ or (ii) $b-t \geq 10$, where r=no. of replications per treatment, b=no. of blocks and t=no. of treatments.

The inaccuracy due to estimating the weights in the balanced incomplete block is also studied by comparing the variance of $\hat{\mu}$ with that of the estimator appropriate when the ratio σ_1^2/σ_2^2 is known.

H. A. David (Blacksburg, Va.)

6648:

Tate, R. F.; and Klett, G. W. Optimal confidence intervals for the variance of a normal distribution. J. Amer. Statist. Assoc. **54** (1959), 674–682.

Tables are given for the computation of confidence intervals for the variance of a normal distribution. The types of intervals are considered, viz. those having minimum length and so called shortest unbiased confidence intervals (introduced by Neyman in 1937). The tables cover five values of confidence coefficients, 0.900; 0.950; 0.990; 0.995; 0.999, and number of degrees of freedom ranging from 2 to 29 in steps of 1 (i.e., sample sizes from 3 to 30 in the single sample case).

J. Janko (Prague)

6649:

Kulldorff, G. A problem of maximum likelihood estimation from a grouped sample. Metrika **2** (1959), 94–99.

Suppose the real line is partitioned into $k+1$ intervals $(-\infty, x_1]$, $(x_1, x_2]$, $\cdots$, $(x_{k-2}, x_{k-1}]$, (x_k, ∞). n observations are made on a random variable X, but the only information available is the interval in which each of the observations falls. Suppose $G(x)$ is a specified c.d.f. and $F(x)$, the c.d.f. of X, satisfies $F(x)=\delta G(x)$, $x \leq x_{k-1}$, where $\delta \geq 0$ is an unknown parameter. The maximum likelihood estimate of δ (which is also the uniformly minimum variance unbiased estimate) is obtained, as is its variance. An extension is given for the case where G also depends on some nuisance parameters. *M. Dwass* (Evanston, Ill.)

6650a:

Kulldorff, Gunnar. Maximum likelihood estimation of the mean of a normal random variable when the sample is grouped. Skand. Aktuarietidskr. **1958**, 1–17 (1959).

6650b:

Kulldorff, Gunnar. Maximum likelihood estimation of the standard deviation of a normal random variable when the sample is grouped. Skand. Aktuarietidskr. **1958**, 18–36 (1959).

In these two papers are examined the problems of estimating the mean and standard deviation, respectively, of a normal population by the method of maximum likelihood, when the data has been grouped. Necessary and sufficient conditions are given for the likelihood function to have a unique root providing an absolute maximum. Large sample consistency and efficiency properties are established. There is a discussion of optimum grouping. *M. Dwass* (Evanston, Ill.)

6651:

Banerjee, Saibal Kumar. Expressions for the lower bound to confidence coefficients. Sankhyā **21** (1959), 127–140.

"A lower bound to the probability of sample estimate plus (and minus) t-times ($t>1$) estimate of sampling error covering the population mean (or total) is derived for samples from non-normal populations. Extensions of the results to the case of ratio estimates and multistage design are also considered.

The paper continues the line of investigation begun by the author in Sankhyā **18** (1957), 391–394 [MR **19**, 1204]." (Author's summary) *G. E. Noether* (Boston, Mass.)

6652:

Gjeddebaek, N. F. Contribution to the study of grouped observations. IV. Some comments on simple estimates. Biometrics **15** (1959), 433–439.

[For part III, see Skand. Aktuarietidskr. **40** (1957), 20–25; MR **20** #367.] The author evaluates the mean squared deviation for estimates of the mean and variance of a normal distribution based on coarsely grouped observations. The "simple" mean and standard deviation estimates with and without Sheppard's corrections are compared with maximum likelihood estimates. Tabulating results for the case where the interval width is twice the standard deviation, the author concludes that the simple estimates with Sheppard's corrections are practically as efficient as the maximum likelihood estimate when the grouping is equidistant (with width no larger than two standard deviations) and with no more than a hundred observations. *H. Chernoff* (Stanford, Calif.)

6653:

Banerjee, D. P. Estimating the parameters of a censored or truncated distribution. Statistica **19** (1959), 211–215.

6654:

★Lehmann, E. L. Testing statistical hypotheses. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xiii+369 pp. $11.00.

This is a thorough and clearly written treatment of the theory of testing statistical hypotheses as formulated in the writings of J. Neyman and E. S. Pearson since 1933, including the latest developments of this theory. The theory of confidence sets and the principal applications of the theories are also included. The author discusses only briefly the closely related problem of the choice between two decisions (which permits one to treat hypothesis and

alternative in a symmetric way), as well as the multi-decision formulations which may be more adequate in certain situations to which the testing of hypotheses approach is traditionally applied. Also essentially omitted are methods based on large-sample considerations, except for a brief outline of the theory of χ^2 tests and likelihood ratio tests.

The book is a thoroughly revised version of the author's lecture notes which appeared in 1949. Parts of the book require a knowledge of the theory of measure in abstract spaces (an outline of which is included). All statistical concepts are developed from the beginning. Each chapter is followed by numerous problems consisting of illustrative examples and further developments of the theory, and by a list of references with informative annotations.

The following list of chapter headings and some selected topics treated in the chapters gives a rough idea of the content of the book: (1) The general decision problem; (2) The probability background (including sufficient statistics, exponential families); (3) Uniformly most powerful tests (including monotone likelihood ratios, confidence bounds, and a proof, partly due to Le Cam, of the optimum property of the sequential probability ratio test); (4) Unbiasedness: Theory and first applications (including similar tests and complete families of distributions); (5) Unbiasedness: Applications to normal distributions; confidence intervals (including permutation tests); (6) Invariance (including rank tests); (7) Linear hypotheses; (8) The minimax principle (including the Hunt-Stein theorem); Appendix (groups, convergence of distributions, dominated families of distributions, the weak compactness theorem).

{Corrections: On p. 162, the "only if" part of Theorem 2 requires an additional condition; see D. Basu, Sankhyā **20** (1958), 223–226 [MR **21** #4494] (communication from the author). There are errors in problem 9, p. 250 (sequential t-test): if the test is based on the sequence $\operatorname{sgn}(x_1), t_1, t_2, \cdots$ (instead of $t_1, t_2, \cdots$), or on $v_1, v_2, \cdots$, where $v_n = x_n/|x_1|$, then the statement in part (iii) of the problem is true (with generalized probability densities).}

W. Hoeffding (Chapel Hill, N.C.)

6655:

★Fix, Evelyn; Hodges, J. L., Jr.; and Lehmann, E. L. The restricted chi-square test. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 92–107. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

The k observations X_i follow a multinomial distribution with expectations np_i. The p_i are known functions of $s<k$ parameters θ_i, and the null hypothesis specifies the values of all except $t<s$ of the θ_i. The restricted chi-square test criterion is

$$U(R) = n \sum_{i=1}^{k} \left[\frac{(R_i - R_i^{**})^2}{R_i^{**}} - \sum_{i=1}^{k} \frac{(R_i - R_i^{*})^2}{R_i^{*}} \right]$$

where $R_i = X_i/n$, and R_i^{**}, R_i^{*} are the "minimum χ^2" estimates of the p_i under the null and alternative hypotheses.

The power function of this test is shown to be a non-central χ^2 with $(s-t)$ d.f. and parameter $\lambda = U(p)$. For the type of alternative hypothesis specified above, the restricted test has the same λ as the ordinary χ^2 test, but the d.f. are reduced from $f=(k-1-t)$ to $f=(s-t)$. For given λ, α the probability $(1-\beta)$ of type II error is roughly proportional to f.

For $f=1(1)6$; $\alpha=.001, .005, .01, .05(.05).30, .40, .50$ the values of λ which make the power $\beta=.5(.1).9, .95$ are tabulated (3 d.p.). These tables supplement the previous tables by E. Fix [Univ. California Publ. Statist. **1** (1949), 15–19; MR **12**, 344].

Methods for computing the restricted chi-square when $s=1, 2$, $t=0$; $s=2$, $t=1$ are given as well as a specific application. *W. G. Cochran* (Cambridge, Mass.)

6656:

MacKay, John H. Asymptotically efficient tests based on the sums of observations. Ann. Math. Statist. **30** (1959), 806–813.

The author defines the asymptotic efficiency of a fixed sample size test of a composite hypothesis in terms of the sample size required to make the maximum risk less than α as $\alpha \to 0$. Extending results of Chernoff derived for simple tests versus simple alternatives [Ann. Math. Statist. **23** (1952), 493–507; MR **15**, 241], the author derives a theorem establishing asymptotic efficiency of certain procedures in certain problems. The theorem seems applicable to those cases where there is an asymptotic analogue of a least favorable distribution which puts all of its probability on two points. It is applied to testing whether the difference of two binomial probabilities ξ and η exceeds δ or is less than $-\delta$.

For m and n trials respectively, the asymptotically efficient test is based on the statistic

$$am(\hat{\xi} - 1/2) + bn(\hat{\eta} - 1/2),$$

where $\hat{\xi}$ and $\hat{\eta}$ are the maximum likelihood estimates and a and b involve m/n and δ.

H. Chernoff (Stanford, Calif.)

6657:

Režný, Zdeněk. On tests of statistical hypotheses by gauging with respect to two limits. Apl. Mat. **4** (1959), 290–302. (Czech. Russian and English summaries)

The following statistical problem is solved: A sample is taken from a population that depends on two parameters, viz. a location and a scale parameter; choose two constants x_1 and x_2, with $x_1 < x_2$, and denote by n_1 the number of sample values falling below x_1 and by n_2 the number of values exceeding x_2. It is required to test hypotheses concerning the location parameter or the scale parameter, using only the information about n_1 and n_2. The author constructs linear functions of n_1 and n_2 on which the optimal decision procedure is to be based and discusses the choice of the limits x_1 and x_2. As the criterion of optimality for his test procedure he uses the relative asymptotic efficiency as defined by Pitman. The results are then applied to the problem of testing location and scale parameters of normal and exponential distributions.

J. Janko (Prague)

6658:

Weber, Erna. Das Ergebnis-Folge-Verfahren (Sequenzanalyse). Grundlagen und Anwendungen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **7** (1957/58), 559–581. (Russian, English and French summaries)

Expository paper reviewing the Neyman-Pearson theory of testing hypotheses and Wald's sequential theory. Two elementary applications are worked out in detail.

D. G. Chapman (Seattle, Wash.)

6659:

Bennett, B. M. On a multivariate version of Fieller's theorem. J. Roy. Statist. Soc. Ser. B **21** (1959), 59–62.

This is a multivariate generalization of Fieller's theorem for determining confidence limits for a ratio α of bivariate normal means. X and Y are sample matrices, each representing n observations on p multivariate normal characters. The population mean values, represented by the vectors ξ and η, are unknown as are the population variance-covariance matrices. α is given by $\eta_i = \alpha\xi_i$ for $i = 1, \cdots, p$. The author introduces $Z_{is} = Y_{is} - \alpha X_{is}$ and applies Hotelling T^2 so as to provide confidence limits for α.

P. S. Dwyer (Ann Arbor, Mich.)

6660:

Teghem, J. L'analyse de la confluence. Cahiers Centre Études Rech. Opér. no. 3 (1959), 5–34.

The method of confluence analysis was developed by Professor R. Frisch more than twenty-five years ago. It dealt with some problems of regression analysis that arise when the ordinary specification for a least squares procedure fails. The present article gives a survey of Frisch's ideas and of the development in subsequent literature on the subject.

T. Haavelmo (Oslo)

6661:

Walter, Edward. Einige Eigenschaften von Symmetrietests. Math. Ann. **137** (1959), 433–453.

Let Ω be a class of probability measures and ω a non-empty subclass of Ω. A class C of tests φ for testing ω against $\Omega - \omega$ is called unbiased if for every measure $P \in \Omega - \omega$ and every level of significance α there is a test $\zeta_{\alpha,P} \in C$ which is strictly unbiased for the simple alternative hypotheses P. Let R_n be the set of all n-tuples of real numbers $(x_1, \cdots, x_n)$. Let K_n be the class of probability measures defined over the Borel sets of R_n which are absolutely continuous with respect to the n-dimensional Lebesgue measure. Let $\Omega_1 \subset K_n$ be the subclass of all product measures, $\Omega_2 \subset K_n$ the subclass of all measures which are invariant under the permutation group of n elements and let $\Omega_3 = \Omega_1 \cap \Omega_2$. Let $\omega_s \subset K_n$ be the subclass of all measures which are symmetric with respect to the origin and $\omega_i = \omega_s \cap \Omega_i$, $i = 1, 2, 3$. It is first proved that the class of tests of Scheffé structure $S(\alpha)$ [cf. H. Scheffé, Ann. Math. Statistics **14** (1943), 227–233; MR **5**, 44] for testing ω_1 against $\Omega_1 - \omega_1$ is unbiased. The completeness of this class is a consequence of theorems of Lehmann and Stein [Ann. Math. Statistics **20** (1949), 28–45; MR **10**, 723]. Further it is proved that the class G of rank tests for testing ω_3 against $\Omega_3 - \omega_3$ is unbiased for $n \geq 3$. This is not true for $n \leq 2$. G is not unbiased for testing ω_1 against $\Omega_1 - \omega_1$.

Let $\mu \geq 1$ be any real number, r an integer such that $1 \leq r \leq n/2$, g_i $(i = 1, \cdots, r)$ non-negative integers with $\sum_{j=1}^{r} g_j > 0$, and m_j $(j = 1, \cdots, r)$ non-negative integers such that $0 = m_0 < m_1 < \cdots < m_r = n + 1$. Suppose further that $\max_{j, g_j > 0} (m_j - m_{j-1} - 1) \geq 2$. For every n-tuple such that $x_i \neq 0$, $x_i \neq x_j$ for $i \neq j$ let $(z_1, \cdots, z_n)$ be the permutation of $x_1, \cdots, x_n$ with $|z_1| < \cdots < |z_n|$. Denote sign z_i by ε_i. Define

$$H(\varepsilon_1, \cdots, \varepsilon_n) = \sum_{j=1}^{r} g_j \Big| \sum_{m_{j-1}+1}^{m_j - 1} \varepsilon_i \Big|^{\mu}.$$

The author shows that the test defined by $\varphi(\varepsilon_1, \cdots, \varepsilon_n) = 1, c, 0$ according as $H(\varepsilon_1, \cdots, \varepsilon_n) >, =, < \gamma$, respectively, where c and γ are determined by the given level of significance, is strictly unbiased for testing ω_3 against $\Omega_3 - \omega_3$. The paper concludes with some considerations for uniformly most powerful rank tests for testing ω_3 against subsets of $\Omega_3 - \omega_3$ [cf. also Ruist, Ark. Mat. **3** (1955), 133–163; MR **16**, 842]. P. 450, line 4 from the top, read $\omega_2' \cap \omega_2$ instead of $\omega_2 \cap \omega_2$.

L. Schmetterer (Hamburg)

6662:

Levis, Vincenzo. Spazi lineari dei minimi quadrati. Rend. Mat. e Appl. (5) **17** (1958), 423–438.

As pointed out by G. Pompilj [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5** (1946), 186–219; MR **9**, 150], locating the linear space of regression (or of least squares) with respect to one given random variable among n such variables is interpretable as decomposing a corresponding Euclidean n-space into a 1-dimensional component space and a complementary $(n-1)$-dimensional component. This author here uses systematic matric notation and notions of Euclidean n-spaces to handle the extension to the linear $(n-r)$-space of regression with respect to a given r-space. He establishes a duality between them and shows that this duality is actually a nondegenerate polarity with respect to an elliptic cone in the n-space. This is familiar in the case $n = 2$.

A. A. Bennett (Providence, R.I.)

6663:

Konijn, H. S. Basing decisions on an analysis of variance. Austral. J. Statist. **1** (1959), 57–68.

This is essentially an expository article which compares the Scheffé and Tukey methods of making multiple comparisons among a set of m means. The Scheffé results are presented in Biometrika **40** (1953), 87–104 [MR **15**, 239, 1139; **16**, 1336]. An excellent summary of various multiple comparison procedures, including these two, is presented by Duncan [Biometrics **11** (1955), 1–42; MR **16**, 842]. Konijn makes the comparisons for a 3×2 factorial experiment. In most factorial experiments, individual comparisons are selected before the experiment is conducted; hence, ordinary confidence limits can be used without the adjustments given in this paper for post-selection of the comparisons to be studied. It should be made clear that the Scheffé and Tukey procedures are concerned with control of experiment-wise error rates, whereas most experimenters contacted by this reviewer are interested in error rates for the comparison or selected comparisons of interest.

Konijn also considers the problem of confidence intervals for multiple means of cells, using the studentized maximum modulus of Pillai and Ramachandran [Ann. Math. Statist. **25** (1954), 565–572; MR **16**, 270].

Some final remarks are made on the use of these methods when one of the factors is considered a random factor.

R. L. Anderson (Raleigh, N.C.)

6664:

Roy, S. N.; and Gnanadesikan, R. Some contributions to ANOVA in one or more dimensions. I. Ann. Math. Statist. **30** (1959), 304–317.

The principal facts of analysis of variance are developed for the univariate case in vector-matrix notation in a form suitable for generalization to the multivariate case. For the fixed effects model conditions are stated for linear functions of the parameters to be estimable, for similar regions to exist for testing linear hypotheses, and for the sums of squares due to several hypotheses to be independent. Simultaneous confidence bounds are given for the non-centrality parameter and the same with any number of its component deviations from hypothesis left out. The random effects model presented is general enough to include the usual connected designs. Necessary and sufficient conditions are stated for the various sums of squares to be independent and distributed as central chi-squares. Simultaneous confidence bounds are given for the ratios of the various components of variance to the error variance. *S. W. Nash* (Vancouver, B.C.)

6665:

Roy, S. N.; and Gnanadesikan, R. Some contributions to ANOVA in one or more dimensions. II. Ann. Math. Statist. **30** (1959), 318–340.

This is the generalization of the paper reviewed above to the multivariate case. For the fixed effects model, the following linear hypothesis is considered:

$$H_0: C(s\times m)\xi(m\times p)M(p\times u) = O(s\times u),$$

where $E[X'](n\times p)=A(n\times m)\xi(m\times p)$ and $X(p\times n)$ is the observation matrix. The random effects model presented is fairly restricted: the covariance matrix of the ith component of variance-covariance is

$$\sum_i (p\times p) = \sigma_i^2 \sum (p\times p),$$

σ_i^2 being a scalar. Necessary and sufficient conditions are stated for the various sums of products matrices to be independent and distributed as central pseudo-Wishart matrices. Simultaneous confidence bounds are given for the σ_i^2. *S. W. Nash* (Vancouver, B.C.)

6666:

Sampford, M. R.; and Taylor, J. Censored observations in randomized block experiments. J. Roy. Statist. Soc. Ser. B **21** (1959), 214–237.

In the present paper the authors discuss the method of deriving the maximum likelihood estimates of the treatment effects and block effects in a randomized block experiment, when some of the observations are censored. The magnitude of the bias in the estimate of the error variance is also investigated. Finally some approximate tests of significance of treatment differences are suggested. These tests are based on the expression for the asymptotic variance-covariance matrix of the estimates of the treatment effects. *R. G. Laha* (Washington, D.C.)

6667:

Youden, W. J.; Connor, W. S.; and Severo, N. C. Measurements made by matching with known standards. Technometrics **1** (1959), 101–109.

Quick estimates of an unknown quantity may be made by matching it against a series of prepared standards with known values. The matching procedure can be considered as a more or less drastic rounding off procedure applied to data with a given standard deviation. Two curves have been prepared to show for various intervals between standards (in terms of multiples of the standard error of measurements, σ). (i) the probability of a successful match, and (ii) the probability that two observers select the same standard.

The paper discusses the problem of choosing the proper size and number of intervals in establishing a standard. Methods of determining σ are also described.

R. L. Anderson (Raleigh, N.C.)

6668:

★Robbins, Herbert. Sequential estimation of the mean of a normal population. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 235–245. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

$\{x_i;\ i=1, 2, \cdots\}$ are NID (μ, σ^2) with both parameters unknown. The problem considered is that of estimating μ sequentially. The estimate is restricted to be $\bar{X}_n$, the mean of the n observations of X that have been observed so that the problem reduces to choosing n. The measure of loss chosen is

$$L = A|\bar{X}_n-\mu|+n.$$

If σ were known $E(L)$ would be minimized when $n_0=(A\sigma/\sqrt{(2\pi)})^{2/3}$. This suggests, when σ is not known, the stopping rule R_1: stop when $N\geqq(AS_N/\sqrt{(2\pi)})^{2/3}$, where $S_N^2=(N-1)^{-1}\sum_{i=1}^{N}(X_i-\bar{X}_N)^2$. It is shown that under this rule

$$E_{R_1}(L) = 2n_0^{3/2}E(N^{-1/2})+E(N).$$

Some empirical determinations of $E_{R_1}(L)$ obtained by Monte Carlo methods are given.

Another stopping rule considered is that associated with sampling observations two at a time and stopping with the first $N=2m+1$ for which (1) is satisfied. Some numerical values of the probability distribution of N under this rule are presented.

D. G. Chapman (Seattle, Wash.)

6669:

Maurice, Rita. A different loss function for the choice between two populations. J. Roy. Statist. Soc. Ser. B **21** (1959), 203–213.

The main content of this paper is the application of Wald's minimax principle to a problem of choosing between two normal populations with the same variance (which is known), when the loss function is given to be of a certain specified form. Some methods of determining the initial sample size under sequential sampling schemes are also discussed. *R. G. Laha* (Washington, D.C.)

6670:

David, Herbert T.; Fay, Edward A.; and Walsh, John E. Acceptance inspection by variables when the measurements are subject to error. Ann. Inst. Statist. Math. Tokyo **10** (1959), 107–129.

"The situation considered is that in which measurement of the characteristic of interest is not exact but subject to

appreciable error. The error is assumed to be unbiased and independent of the actual value of the characteristic measured. The population and error variances, σ^2 and σ_e^2, are assumed to be such that σ/σ_e has a known lower limit which is greater than zero. The probability distributions involved are assumed to be normal, while the actual values and measurement errors each form a random sample. For suitable specified acceptable and unacceptable fractions defective, and for σ_e assumed known and unknown, this paper presents one-sided acceptance inspection criteria which are optimum in a specified sense, and which have the property that the producer's and consumer's risks have specified upper bounds." (Authors' summary) *D. M. Sandelius* (Åmål)

6671:

Box, G. E. P.; and Hunter, J. S. Condensed calculations for evolutionary operation programs. Technometrics **1** (1959), 77–95.

The authors describe and give the theoretical background of a simple routine for successive recalculation of interval estimates of factorial effects in cyclically extended experiments, where in each cycle a complete set of the factorial combinations is run. *D. M. Sandelius* (Åmål)

6672:

Aitchison, J. A statistical theory of remnants. J. Roy. Statist. Soc. Ser. B **21** (1959), 158–168.

A manufacturer produces lengths, X, of cloth which, cut into as many units as possible, leave a remnant of length Y. There is insufficient control to make sure that each piece is an integral multiple of the unit length. The author finds the probability density of Y given that of X. Let X be normally distributed with mean μ, standard deviation σ, and denote $\mu-[\mu]$ by λ. If $\sigma \geqq 1$, the average value of Y is .5, and no better control is possible. But, for a fixed σ, $\sigma<1$, there exists a unique value of λ_0 which minimizes $E(Y)$. Based on these results, the author provides two tests to determine optimum control conditions. The results are extended to non-normal distributions and to k-dimensional remnants, $k \geqq 2$.

L. A. Aroian (Los Angeles, Calif.)

6673:

Cusimano, Giovanni. Il metodo di Montecarlo e le sue applicazioni nella statistica industriale. Statistica **19** (1959), 216–229.

6674:

Steck, G. P.; and Owen, D. B. Percentage points for the distribution of outgoing quality. J. Amer. Statist. Assoc. **54** (1959), 689–694.

"When a consumer buys only a few lots from a producer according to some scheme based on the results of sampling inspection, he should not depend on the average outgoing quality limit as being the upper bound on the defectiveness of the lots he buys. The consumer should, in fact, examine the percentage points of the distribution of outgoing quality. Approximate percentage points for the distribution of outgoing quality for a finite number of lots are found in this paper." (From the author's summary)

L. A. Aroian (Los Angeles, Calif.)

6675:

★Rosenblatt, M. Statistical analysis of stochastic processes with stationary residuals. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 246–275. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

The weakly (second order) stationary vector process $x(t)$ with mean function zero is assumed to have a spectral density which is estimated by weighted averages of the periodogram. Such estimates (spectrograph estimates) have asymptotic properties similar to those of quadratic form estimates. Proofs are sketched for typical special cases. Bias and conditions for asymptotic normality of spectrograph estimates are investigated. After summarizing and extending his recent work on the estimation of regression coefficients [Proc. 3rd Berkeley Sympos. Math. Statist. Prob., 1954/55, vol. I, pp. 165–186, Univ. of Calif. Press, Berkeley-Los Angeles, 1956; MR **18**, 948], the author points out that his results concerning spectrograph estimates remain valid, with simple modifications, when the mean function (regression) is a polynomial.

F. L. Spitzer (Minneapolis, Minn.)

6676:

Balakrishnan, A. V. On a characterization of covariances. Ann. Math. Statist. **30** (1959), 670–675.

The following theorem is proved. Let

$$R(t) = \int_{-\infty}^{+\infty} \exp(itx)dG(x)$$

be a continuous stationary covariance on $(-\infty, +\infty)$. Then $R(s,t)=R(s-t)-m(s)\bar{m}(t)$ is a covariance if and only if $m(t)=\int_{-\infty}^{+\infty}\exp(itx)dM(x)$, where M is a function of bounded variation such that $\int |dM/dG|^2 dG \leq 1$. The discrete form of the theorem is then used to obtain necessary and sufficient conditions for consistency (in the mean square sense) of linear unbiased estimates of mean values of second order sequences. *M. Loève* (Berkeley, Calif.)

NUMERICAL METHODS

See also 6351, 6372a–c, 6408, 6453, 6673, 6715, 7017.

6677:

Page, E. S. Pseudo-random elements for computers. Appl. Statist. **8** (1959), 124–131.

An expository paper which reviews procedures for generating pseudo observations within a digital computer. Using simple numerical examples the author lucidly demonstrates how pseudo random numbers are generated by "Middle of the square", "Multiplicative congruential" and "Additive congruential" methods. The paper concludes with some general remarks for generating pseudo variates from arbitrary distributions. Methods related to the generation of variates from the normal distribution are given specific attention. *M. E. Muller* (New York, N.Y.)

6678:

Green, Bert F., Jr.; Smith, J. E. Keith; and Klem, Laura. Empirical tests of an additive random number generator. J. Assoc. Comput. Mach. **6** (1959), 527–537.

An approach to generate pseudo random numbers within a digital computer is to use an additive congruence relation. For example, with a binary machine the jth number in the sequence could be determined by

$$x_j = (x_{j-1}+x_{j-n}) \bmod 2^r, \quad j > n, \tag{1}$$

where $0 \leq x_j < 2^r$, and where r is the number of bits used to specify each fixed point number in the machine. When $n=2$, (1) has been explored by several authors, for example, Taussky and Todd [Sympos. Monte Carlo Methods, Univ. of Florida, 1954, pp. 15–28, Wiley, New York, 1956; MR 18, 237]. Based on the statistical tests performed to date, it appears that the procedure will give rise to numbers that: (1) may be accepted as having come from a rectangular distribution, (2) possess significant sequential dependence. To avoid this type of dependence several authors suggested that only every other number which is generated be utilized. Another avenue is to explore the more general case of $n>2$.

For $n>2$, the authors have tested samples when (1) $r=15$ for $n=5$, 16, 32 and (2) $r=35$ for $n=6$, 10, 13, 15, 16. This procedure requires that the computer program initially have access to n "random numbers" and subsequently have access to the last n numbers generated. (Depending on the computer, this requirement can vitiate any possible saving in computing time that an additive congruence procedure may have over a multiplicative congruence approach.) Using the samples studied, the authors indicate that serial dependence can be significant if n is less than sixteen. *M. E. Muller* (New York, N.Y.)

6679:

Motoo, Minoru. Some evaluations for continuous Monte Carlo method by using Brownian hitting process. Ann. Inst. Statist. Math. Tokyo **11** (1959), 49–54.

The similarity of the results presented in this paper with some of those obtained by the reviewer [Ann. Math. Statist. **27** (1956), 569–589; MR **19**, 580] may be best stated by quoting from the Introduction to the paper, namely "In this paper, we want to give an estimation of the solution of Dirichlet's problem by Monte Carlo method (which is essentially the same as in Muller's paper)". When the domain of the problem is convex, the reviewer gave an asymptotic upper bound to the mean number of steps required for termination by a truncated process. (This upper bound is a linear function of the dimensionality of the space.) Following this approach, using a m-dimensional Brownian hitting process, and using the explicit form of Green's function for the m-sphere, the author provides the following interesting upper bound:

$$(1+\log p(x)/\delta)16m,$$

where, following the notation of the reviewer, the process is truncated when it is within distance δ of the boundary, and where the solution to the problem is desired at point x which is at distance $p(x)$ from the boundary.

M. E. Muller (New York, N.Y.)

6680:

Sugiyama, Hiroshi; and Miyatake, Osamu. Design of "Random Walker" for Monte-Carlo method. I. Theory. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 35–41.

A heuristic presentation is given for the mathematical basis of certain Monte Carlo procedures which are suggested to estimate the solution of problems such as (1) eigenvalues, (2) Dirichlet's problem, (3) Poisson's equation. The authors appear to be unaware that the mathematical justification for the use of random walk procedures in numerical analysis has an extensive literature, for example, J. H. Curtiss [Proc. Sem. Sci. Comput. 1949, pp. 87–109, IBM Corp., New York, 1950; MR **13**, 286], W. Wasow [J. Res. Nat. Bur. Standards **46** (1951), 462–471; MR **13**, 960], M. E. Muller [J. Math. Phys. **35** (1956), 89–113; MR **17**, 1093]. *M. E. Muller* (New York, N.Y.)

6681:

Stiefel, E. Über diskrete und lineare Tschebyscheff-Approximationen. Numer. Math. **1** (1959), 1–28.

The problem at hand is to find a point $x=(x_1, \cdots, x_m)$ such that the maximum of the absolute values of the numbers $h_j=a_{jk}x_k+c_j$ $(j=1, \cdots, n;\ k=1, \cdots, m<n)$ is a minimum. Such a point is called a Tschebyscheff point. It is shown that if the m-rowed minors of the matrix (a_{jk}) have rank m there is a unique such point. The proof is constructive in nature and yields an effective computational method for finding this point; extensions of this result and connections with the least square solution are given. The results are applied to the problem of finding polynomial approximations in one or more variables. Finally, the results are applied to that of approximating a function $f(x)$ by a linear combination of functions $\varphi_1(x), \cdots, \varphi_m(x)$. In this connection a theory of s-functions is developed. Numerical examples of the theory and of computational techniques are given.

M. R. Hestenes (Los Angeles, Calif.)

6682:

★Stiefel, Eduard L. Numerical methods of Tchebycheff approximation. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 217–232. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

The present paper is an expository paper describing in some detail the theoretical and numerical aspects of a linear Tschebyscheff approximation of a function $f(x)$ by given functions $\varphi_1(x), \cdots, \varphi_m(x)$ at specified points $x_1, \cdots, x_n$. The theoretical development is constructive in nature and leads to computational techniques for solving the problem numerically. Connections with least square and least qth approximations are given. The special case of polynomial approximations is discussed. Alternate computational procedures are described, giving their advantages and disadvantages. Several open problems of computational significance are described.

M. R. Hestenes (Los Angeles, Calif.)

6683:

von Targonski, Georg. Interpolation durch Reihen iterierter Funktionen. Soc. Sci. Fenn. Comment. Phys.-Math. **20** (1957), no. 9, 12 pp.

Let $g_0(x)=x$, $g_n(x)=g(g_{n-1}(x))$, $n=1, 2, \cdots$, be the iterates of a real function $g(x)$. This paper investigates the following interpolation problem: Given a sequence

α_n ($\alpha_0 \neq 0$) with $g_n(\alpha_{n-1}) = 0$ ($n = 1, 2, \cdots$), find a continuous function of the form

$$F(x) = \sum_{n=0}^{\infty} \vartheta_n g_n(x) \tag{1}$$

such that $F(\alpha_n) = \xi_n$ with given real numbers ξ_n. The special choice of the α_n implies that

$$\vartheta_n = (1/\alpha_0)[\xi_n - (\vartheta_0\alpha_n + \cdots + \vartheta_{n-1}\alpha_1)].$$

Thus the problem consists mainly in proving the absolute and uniform convergence of the series (1). For functions of the form $g(x) = h[ch^{-1}(x)]$ (i.e., $g_n(x) = h[c^n h^{-1}(x)]$) the following result is obtained: Assume that (i) $|h(x)| < \min(|x|, M)$ for $0 \leq x < \infty$, (ii) $a_0 > 0$ exists with $g(a_0) = 0$, (iii) $h(x)$ is continuous, positive and strongly monotonically increasing for $0 < x \leq a_0$; define $\alpha_n = g^{-1}(\alpha_{n-1})$, $n = 1, 2, \cdots$, and let $\xi_n = O(1/R^n)$ with some $R > 1$. Then the series (1) converges absolutely and uniformly for $0 \leq x \leq a_0$ and $F(\alpha_n) = \xi_n$ ($n = 0, 1, \cdots$).

W. Rheinboldt (Syracuse, N.Y.)

6684:

Hornecker, Georges. Approximations rationnelles voisines de la meilleure approximation au sens de Tchebycheff. C. R. Acad. Sci. Paris **249** (1959), 939–941.

Partant du fait qu'une fraction homographique a un développement très simple en polynômes de Tchebycheff, l'auteur écrit des approximations de fonctions dont on suppose connu le développement en polynômes de Tchebycheff.

Ces approximations sont formées de termes polynomiaux suivis d'une ou deux fractions dont les dénominateurs sont de degré 1 ou 2. Appliqué à e^x, entre -1 et $+1$, ceci donne une erreur inférieure à $2{,}6 \cdot 10^{-7}$ avec 4 multiplications ou divisions. *J. Kuntzmann* (Grenoble)

6685:

Hofmann, Rudolf. Bemerkung zur Berechnung spezieller Potenzsummen. Z. Angew. Math. Phys. **10** (1959), 416–420. (English summary)

The author transforms a certain class of finite series whose terms were originally functions of known Laplace transforms. Closed forms for the sum are sometimes obtained, e.g., the series $\sum_{k=1}^{n} \sin^M kx$, $\sum_{k=1}^{n} \cos^M kx$ and $\sum_{k=1}^{n} (-1)^k k^M$, where M is a positive integer.

N. S. Mendelsohn (Winnipeg, Man.)

6686:

★Яковкин, М. В. Численная теория приводимости многочленов. [Yakovkin, M. V. Numerical theory of the reducibility of polynomials.] Akademiya Nauk SSSR, Vyčislitel'nyĭ Centr. Izdat. Akad. Nauk SSSR, Moscow, 1959. 135 pp. 5 rubles.

The first two chapters summarize reducibility criteria, the third states a theorem and describes a method previously published by the author [Dokl. Akad. Nauk SSSR **93** (1953), 629–631, 783–785; MR **15**, 790]. The theorem is as follows. Let $f(x)$, $\phi(x)$ and $\psi(x)$ be polynomials whose coefficients are non-negative integers. Then $f(x) = \phi(x)\psi(x)$ if and only if $f(1) = \phi(1)\psi(1)$ and $f(t) = \phi(t)\psi(t)$, where t is any integer exceeding every coefficient of f. A natural choice is $t = 10^m$ for a suitable m. Any polynomial with integral coefficients can be replaced by one whose coefficients are non-negative integers by applying a linear substitution in x. Hence the problem is reduced to that of examining the factors of integers.

A final chapter discusses the reduction of polynomials in several variables. *A. S. Householder* (Oak Ridge, Tenn.)

6687:

★Eybert, R. Sur la sommation pratique des séries. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 39–42. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii + 87 pp. 1800 francs.

Pour sommer la série divergente ou peu convergente

$$a_0 + a_1x + a_2x^2 + \cdots,$$

l'auteur multiplie par $(a_0 - a_1x)(a_1 - a_2x)\cdots$ et somme la série trouvée arrêtée à son terme minimum.

Il applique à la série connue

$$1 - 1 + 1 \cdot 2 - 1 \cdot 2 \cdot 3 + \cdots.$$

J. Kuntzmann (Grenoble)

6688:

Picone, Mauro. L'automazione del calcolo e il progresso dell'analisi matematica. Ricerca Sci. **28** (1958), 697–717. (French, English and German summaries)

This paper begins with a discussion of contributions of automatic computers to the development of mathematical analysis; then discusses the invertibility of matrices in terms of certain particular matrix norms. The author was apparently unaware of papers by Ostrowski [Math. Z. **63** (1955), 2–18; MR **17**, 228] and by Werner Gautschi [Compositio Math. **12** (1954), 1–16; MR **16**, 326; and others]. The property mainly exploited is that, for large ν, $\|A^\nu\|^{1/\nu}$ approaches the spectral radius of A.

A. S. Householder (Oak Ridge, Tenn.)

6689:

Wilf, Herbert S. Matrix inversion by the annihilation of rank. J. Soc. Indust. Appl. Math. **7** (1959), 149–151.

The author recommends a certain decomposition for inverting a matrix by the method of Sherman, et al. [See the reviewer's *Principles of numerical analysis*, McGraw-Hill, New York, 1953; MR **15**, 470; p. 79.] The decomposition is most easily represented by considering a partitioning

$$\begin{pmatrix} \alpha & a' \\ a & A \end{pmatrix} = \alpha^{-1}\begin{pmatrix} \alpha \\ a \end{pmatrix}\begin{pmatrix} \alpha & a' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A - a\,a'/\alpha \end{pmatrix}$$

(a' is a row vector, but not necessarily the transpose of a). By continuing, the given matrix is expressed as a sum of at most n matrices of rank 1, symmetric if the original matrix is symmetric. Hence, if this matrix differs from the matrix to be inverted by a nonsingular diagonal matrix, application of the method of Sherman inverts the matrix in n steps, and symmetry is retained if present at the start. There are several misprints.

A. S. Householder (Oak Ridge, Tenn.)

6690:

de Greiff Bravo, Luis. Matrix inversion. Rev. Acad. Colombiana Ci. Exact. Fis. Nat. **10** (1959), 295–299. (Spanish)

The author discovers (and demonstrates) that if $E_k \cdots E_2E_1A = I$, then $E_k \cdots E_2E_1 = A^{-1}$, all matrices

being square; thence that matrices E_i can be selected which, alternately, differ from I by a single diagonal element, and by a matrix of rank one.

A. S. Householder (Oak Ridge, Tenn.)

6691:

Ostrowski, A. M. On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. V. Usual Rayleigh quotient for non-Hermitian matrices and linear elementary divisors. Arch. Rational Mech. Anal. **3**, 472–481 (1959).

This is the fifth of a series of papers [for part IV, see same Arch. **3** (1959), 341–347; MR **21** #4541b] studying the convergence of the Rayleigh quotient iteration for finding characteristic roots and characteristic vectors of a matrix A. In the present paper A is assumed to be non-Hermitian. The iteration studied is given by the formulas

$$\xi_\nu = (A-\lambda_{\nu-1}I)^{-1}\xi_{\nu-1}, \quad \lambda_\nu = \frac{\xi_\nu^* A\xi_\nu}{\xi_\nu^*\xi_\nu} \quad (\nu=1, 2, 3, \cdots).$$

Here the usual Rayleigh quotient is used, whereas in an earlier paper a generalized Rayleigh quotient was used. It is shown that there is a number δ_0 depending only on A such that if η is a characteristic vector and $|\xi_0-\eta| < \delta_0|\eta|$, then λ_ν converges to the characteristic values corresponding to η. If the elementary divisors of A are linear, then in general the convergence of λ_ν to σ is quadratic. Further theorems on the rate of convergence are given.

M. R. Hestenes (Los Angeles, Calif.)

6692:

Nitsche, Joachim A. Einfache Fehlerschranken beim Eigenwertproblem symmetrischer Matrizen. Z. Angew. Math. Mech. **39** (1959), 322–325. (English, French and Russian summaries)

Given the n orthonormal vectors y_i, and the orthonormal set x_i of characteristic vectors of A, let $\mu_i = y_i'Ay_i$, $\varepsilon_i = [y_i'A^2y_i - (y_i'Ay_i)^2]^{1/2}$. Then if

$$q = \min|\mu_i-\mu_k|, \quad \varepsilon = \max\varepsilon_i,$$

the inequality $q > 2\varepsilon n^{1/2}$ implies

$$(\mu_i-\lambda_i) \leq \varepsilon n^{1/2}, \quad \|y_i-x_i\| \leq \eta(1+\eta),$$

where $\eta = \varepsilon/[q-\varepsilon n^{1/2}]$. For the characteristic roots λ_i the Weinstein theorem [Proc. Nat. Acad. Sci. U.S.A. **20** (1934), 529–532] is much sharper, but for the vectors almost nothing has been published.

A. S. Householder (Oak Ridge, Tenn.)

6693:

Baker, George A., Jr. A new derivation of Newton's identities and their application to the calculation of the eigenvalues of a matrix. J. Soc. Indust. Appl. Math. **7** (1959), 143–148.

The new results obtained in this paper are not mentioned in the title. These are expressions for the standard deviations of errors generated in the computation, obtained from fairly reasonable hypotheses.

A. S. Householder (Oak Ridge, Tenn.)

6694:

Akaike, Hirotugu. On a successive transformation of probability distribution and its application to the analysis of the optimum gradient method. Ann. Inst. Statist. Math. Tokyo **11** (1959), 1–16.

Let A, P be nonsingular n-by-n matrices, with P positive definite, and let x, b be n-rowed column vectors. In the optimum gradient method of solving the system $Ax=b$, one picks x_0 arbitrarily, and (for $k=1, 2, \cdots$) chooses γ in $z=x_k-\gamma\zeta_k$ so as to minimize $(Az-b, P(Az-b))$, where $\zeta_k=2A'P(Ax_k-b)$. The minimizing vector is taken as x_{k+1}. Let $\varepsilon_k=x_k-A^{-1}b$. Let the eigenvalues of $A'PA$ be $\lambda_1<\lambda_2<\cdots<\lambda_n$, with eigenvectors $\xi_1, \xi_2, \cdots, \xi_n$. Assume $(\varepsilon_0, \xi_1)\neq 0$ and $(\varepsilon_0, \xi_n)\neq 0$. The author proves the following conjecture of Forsythe and Motzkin [abstract, Bull. Amer. Math. Soc. **57** (1951), 183, 304–305]: (i) As $k\to\infty$, $\varepsilon_k/|\varepsilon_k|$ tends to the plane π spanned by ξ_1 and ξ_n; (ii) $\varepsilon_k/|\varepsilon_k|$ alternates asymptotically between two directions in π.

In the proof the author represents ε_k as $\sum_{i=1}^n \alpha_i^{(k)}\lambda_i^{-1}\xi_i$, and defines a corresponding probability distribution $\mathbf{P}^{(k)}$ on $\{\lambda_i\}$ with the value $[\alpha_i^{(k)}]^2/\sum_i[\alpha_i^{(k)}]^2$ at λ_i ($i=1, \cdots, n$). He defines $\mathbf{P}^{(k+1)}=T\mathbf{P}^{(k)}$ as the corresponding distribution for ε_{k+1}. He derives expressions for the mean and variance of $T\mathbf{P}^{(k)}$ in terms of those of $\mathbf{P}^{(k)}$. Since the variance of $T\mathbf{P}^{(k)}$ is found to be $\geq$ the variance of $\mathbf{P}^{(k)}$, he is able to prove the convergence of a subsequence of $\{\mathbf{P}^{(k)}\}$ to a distribution $\mathbf{P}^{(\infty)}$ whose variance equals that of $T\mathbf{P}^{(\infty)}$. Finally, he shows that $\mathbf{P}^{(\infty)}$ can have only two points in its spectrum, and out of this can prove the theorem. {It is perhaps worth remarking that a similar proof of Akaike's theorem was presented by R. J. Arms to the Houston meeting of the Association for Computing Machinery, June 19, 1957, but has apparently not been published.}

The author gives other theorems on the rate of convergence of the optimum gradient method, together with a discussion of an acceleration process like that of A. I. and G. E. Forsythe [*Contributions to the solution of systems of linear equations and the determination of eigenvalues*, pp. 55–69, Nat. Bur. Standards Appl. Math. Ser. No. 39, U.S. Govt. Printing Office, Washington, D.C., 1954; MR **16**, 630]. There are data from numerical experimentation with the acceleration for a matrix of order 6 taken from last paper cited.

G. E. Forsythe (Stanford, Calif.)

6695:

Christeller, Silvio. Ermittlung der Wirzeln einer Gleichung 4. Grades durch Aufspalten in quadratische Faktoren. Z. Angew. Math. Phys. **10** (1959), 525–527. (English summary)

The roots of a quartic polynomial with real coefficients can readily be obtained provided one quadratic factor is known. In this paper a procedure is described for determining such a factor with an improvement process based on the Newton-Raphson method rather than on linear interpolation. It is, however, restricted to quartics whose cubic term is missing.

C. Froese (Renton, Wash.)

6696:

Chao, F. H. Routh's method for finding complex roots. Acta Math. Sinica **9** (1959), 101–113. (Chinese. English summary)

"Given a polynomial with real coefficients, it is well known that Routh's method can be used to determine the number of complex roots of the given polynomial in the

right half-plane. In this paper, the author proves that the imaginary part of the pair of conjugate imaginary roots with the largest real part can also be found by Routh's method. When we have a pair of roots, real or complex, in the neighborhood of the origin, Routh's method gives in general better results than roots determined by the tail quadratic form of the given polynomial. When the imaginary part of the roots is not small, linear interpolation can be used to determine a better approximation. Finally, using Routh's method, a method for determining the number of imaginary roots is devised." (Author's summary) *Ky Fan* (Notre Dame, Ind.)

6697:

Bartholomew, Gerald E. Numerical integration over the triangle. Math. Tables Aids Comput. **13** (1959), 295–298.

The author presents a method of calculating evaluation points for numerical integration over a triangle, using successive decompositions. The method involves calculating for a fixed triangle the evaluation points using the midpoints of the sides [Hammer, Marlowe, and Stroud, Math. Tables Aids Comput. **10** (1956), 130–137; MR **19**, 177]. A linear transformation transfers the points to any triangle. A flow diagram of the program is included and results of trial calculations are given. The use for the method is presumed to be primarily for triangulated polygonal regions. *P. C. Hammer* (Madison, Wis.)

6698:

Rapp, F. Unbestimmte Integrale, die Potenzen von Zylinderfunktionen enthalten. Z. Angew. Math. Mech. **39** (1959), 328.

6699:

O'Brien, J. P. Note on decomposition into first order of multi-order linear differential equations with constant coefficients. Comput. J. **2** (1959), 144.

"This note is a reminder of an implicit method of dealing with the derivatives of the forcing function when decomposing a multi-order system into a set of first-order differential equations." *Author's summary*

6700:

Neidhofer, Gerhard. Intégration approchée des équations différentielles lorsque la dérivée d'ordre le plus élevé ne figure que dans un terme correctif. Chiffres **2** (1959), 43–53. (English, German and Russian summaries)

L'équation $hy''+ay'+by=0$ (h petit) est intégrée par l'itération

$$hy_n''+ay_{n+1}'+by_{n+1}=0.$$

La convergence est étudiée pour le cas où les coefficients sont constants.

Divers exemples montrent qu'il peut y avoir convergence numérique, malgré la divergence du procédé.

J. Kuntzmann (Grenoble)

6701:

Mitchell, A. R. The influence of critical boundary conditions on finite difference solutions of two point boundary value problems. Math. Tables Aids Comput. **13** (1959), 252–260.

The author studies the numerical solution of the differential equation $y''+pk^{1/2}y'+ky=0$, subject to several two point boundary conditions. A centered second difference equation is used, and the exact solutions of the differential and difference equations are compared. If $p^2<4$, no solutions of the differential equation and the difference equation exist for certain values of k, and these critical values are not the same for the differential and difference problems. He shows that the relative error in the solution of the difference equation is quite large for k in the neighborhood of these critical values. He also discusses the convergence of the solution of the difference equation as the mesh size is reduced. He produces an example in which a critical value of the difference problem behaves in a very peculiar fashion as the grid size is reduced. For a while the critical value diverges strongly from that of the differential equation and then it finally converges to it. *J. Douglas, Jr.* (Houston, Tex.)

6702:

★Douglis, Avron. On discontinuous solutions of nonlinear, first order, partial differential equations. Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 24–31. Libreria Eredi Virgilio Veschi, Rome, 1959. xii+158 pp.

This paper is a summary without proof of a paper yet to appear treating the numerical solution of a scalar conservation law $u_t+(F(u))_x=0$, where $F(u)$ is uniformly convex and is continuously differentiable. The author discusses the implicit difference equation

$$(\Delta t)^{-1}(U_{ij}-U_{i,j-1})+F(\Delta x)^{-1}(U_{ij}-U_{i-1,j}))=\varepsilon\Delta x\sum_{m=0}^{i}p_{mj},$$

where U_{ij} is the approximation of the integral of u with respect to x. The terms p_{mj} are tempering factors and may be chosen in a variety of ways, subject to certain restrictions listed in the paper. Included as special cases are both the linear viscosity method and the method of von Neumann and Richtmyer. The author promises proof that the solution of the difference system for the initial value problem converges to a weak solution of the differential equation, provided that $\Delta t/\Delta x$ is smaller than a bound depending on the convexity of $F(u)$.

J. Douglas, Jr. (Houston, Tex.)

6703:

★Lax, Peter D. On difference schemes for solving initial value problems for conservation laws. Symposium on questions of numerical analysis: Proceedings of the Rome Symposium (30 June–1 July 1958) organized by the Provisional International Computation Centre, pp. 69–78. Libreria Eredi Virgilio Veschi, Rome. vii+79 pp.

The author introduces a very general difference method for approximating possibly discontinuous solutions of systems of first order, hyperbolic differential equations of the form $u_t=(f(u))_x$, where u is a vector. Let $g(u_1, \cdots, u_{2p})$ be a vector-valued function of $2p$ vector arguments subject to the restriction $g(u, \cdots, u)=f(u)$. Let $g(x, t)=g(u_1, \cdots, u_{2p})$, where $u_j=u(x+(j-p-\frac{1}{2})\Delta x, t)$. Then the

author's difference equation is $u(x, t+\Delta t)=u(x, t)+\Delta t(\Delta x)^{-1}(g(x+\frac{1}{2}\Delta x, t)-g(x-\frac{1}{2}\Delta x, t))$. He shows for the initial value problem that, if the solution of the difference equation tends boundedly and almost everywhere to a limit, the limit is a weak solution of the original system. He discusses the choice of g so as to obtain higher order accuracy. Finally, he demonstrates convergence for a special case. The method is a generalization of an earlier method of the author [Comm. Pure Appl. Math. **7** (1954), 159–193; MR **16**, 524]. *J. Douglas, Jr.* (Houston, Tex.)

6704:

★**John, Fritz. Numerical solution of problems which are not well posed in the sense of Hadamard.** Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 103–116. Libreria Eredi Virgilio Veschi, Rome, 1959. xii + 158 pp.

It is pointed out that most problems of physical interest are not well-posed in the sense of Hadamard, since the data are neither completely known nor perfectly accurate. The approximate solution of a large class of not-well-posed problems is formulated in a general setting. The error bound depends both upon the approximation scheme being used and upon the error in the data. The latter puts an inherent limitation on the accuracy of any approximation scheme.

The problem is called well-behaved if the error due to an inaccuracy δ in the data is of the order of a positive power of δ. Examples are given of both well-behaved and not-well-behaved not-well-posed problems. It is pointed out that the error made in approximating the solution of a well-behaved not-well-posed problem by means of a fixed finite difference scheme with optionally chosen mesh size h is of an order larger than any positive power of δ, since such a scheme is unstable for small h. Thus in order to obtain an error of an order of a power of δ one must let the finite difference scheme itself depend upon δ.

H. F. Weinberger (College Park, Md.)

6705:

Greenstadt, J. On the reduction of continuous problems to discrete form. IBM J. Res. Develop. **3** (1959), 355–363.

The problem of approximating a function u which minimizes a certain functional $F(u)$ is considered. The domain of definition of u is divided into a finite number of cells on each of which u is approximated by a function depending on a finite number of parameters. The functional is extended to allow for discontinuities along the cell walls. It is then minimized with respect to the parameters (Ritz method). Assumptions on F, convergence, and degree of approximation are not discussed.

H. F. Weinberger (College Park, Md.)

6706:

Douglas, Jim, Jr.; and Gallie, T. M., Jr. An approximate solution of an improper boundary value problem. Duke Math. J. **26** (1959), 339–347.

Here, the following problem is discussed: u is a function about which nothing is known except that on the half plane $y \geq 0$ it is bounded and harmonic and that on a certain line $y=c$, $c>0$, boundary data are given. It is assumed that the difference between the boundary data and the true values of u is small. A method is then described for finding a function v which is a solution of the differential equation and is an arbitrarily good approximation to the unknown function u on an arbitrary compact subset R of the half plane $y>0$. A sequence of approximating functions, whose values it is possible to compute, are specified and it is shown that they converge uniformly on R to u. The only evidence that the rate of convergence is rapid is that obtained from numerical experiments.

M. Lister (University Park, Pa.)

6707:

Strang, W. Gilbert. On the order of convergence of the Crank-Nicolson procedure. J. Math. Phys. **38** (1959/60), 141–144.

The author discusses the Crank-Nicolson difference equation for the parabolic equation $u_t=u_{xx}+d(x, t)$. If the solution of the first boundary problem is sufficiently smooth, the solution of the difference equation converges point-wise with an error that is $O((\Delta x)^2)$ if $\Delta t=O(\Delta x)$. The proof makes use of the explicitly known eigenfunctions of the process. Similar results have been obtained by M. Lees by different methods.

J. Douglas, Jr. (Houston, Tex.)

6708:

Birger, I. A. On a method of solution of integral equations with large values of the parameter. J. Appl. Math. Mech. **22** (1958), 1149–1152 (808–809 Prikl. Mat. Meh.).

The author describes a method of solving integral equations of the form $y=\lambda Ky+f$. He assumes that f can be expanded as a series $f=\sum_{n=1}^{\infty} c_n y_n$ in terms of the characteristic functions (y_n) of the homogeneous equation, the coefficients being given by

$$c_n = (f, y_n) = \int_a^b f y_n h\,dx,$$

where h is a weight function with respect to which the (y_n) are orthonormal. Let $|\lambda_{j-1}|<|\lambda|<|\lambda_j|$ and suppose that $\lambda_1, \cdots, \lambda_{j-1}$ and $y_1, \cdots, y_{j-1}$ can be found. Write

$$K_j y = Ky - \sum_{n=1}^{j-1} y_n (Ky, y_n).$$

The modified equation $y=\lambda K_j y+f$ can then be solved by the usual iterative process; let y^* be its solution. The original equation then has the solution

$$y = y^* + \sum_{n=1}^{j-1} \frac{y_n}{(\lambda_n/\lambda)-1}(f, y_n).$$

F. Smithies (Cambridge, England)

6709:

Wynn, P. On the propagation of error in certain non-linear algorithms. Numer. Math. **1** (1959), 142–149.

The propagation of errors in the $q-d$ algorithm, the first and second q-algorithm, the η-algorithm and the ρ-algorithm is investigated by means of a linearization of the defining relation between the elements of the iteration.

It is shown that the $q-d$ algorithm is unstable, though

not explosive, and expressions for the condition numbers in the other algorithms are given.

R. W. Hamming (Murray Hill, N.J.)

6710:

Radó, Francisc. Équations fonctionnelles en liaison avec la nomographie. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 249–319. (Romanian. Russian and French summaries)

Dies ist eine sehr inhaltsreiche interessante Arbeit (die Dissertation des Verfassers) über Verwendung von Funktionalgleichungen in der Nomographie und über ihren Zusammenhang mit der Theorie der Gewebe.

Nach einer Einleitung bezüglich Grundbegriffe der Nomographie [etwa im Sinne von M. d'Ocagne, *Traité de nomographie*, 2nd. ed., Gauthier-Villars, Paris, 1921] und bezüglich der Assoziativitäts- und Bisymmetriegleichungen [J. Aczél, Bull. Amer. Math. Soc. **54** (1948), 392–400; Bull. Soc. Math. France **76** (1948), 59–64; MR **9**, 501; **10**, 685; M. Hosszú, Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 335–342; MR **15**, 203] führt der Verf. die Thomson-, Reidemeister- und Sechseck- (oder Brianchon-) Bedingungen der Geometrie der Gewebe [vgl. etwa W. Blaschke und G. Bol, *Geometrie der Gewebe*, Springer, Berlin, 1938] vor, zur Charakterisierung der in beiden Theorien grundlegenden "Pseudosummen"

$$(*) \qquad F(x, y) = h(f(x)+g(y))$$

(f, g, h sind stetig und strengmonoton) unter den strengmonotonen stetigen Funktionen. Er leitet nämlich aus (*) alle drei obengenannten Bedingungen und unter Monotonie- und Stetigkeitsvoraussetzungen aus der letzten, schwächsten Bedingung (*) ab. Er zeigt, dass die zwei ersteren Bedingungen mit der Kommutativität bzw. Assoziativität der Funktionen

$$G(x, y) = F(F_1(u, x), F_2(y, v))$$

für alle u, v äquivalent sind, wo $x = F_1(y, z)$ und $y = F_2(z, x)$ die als eindeutig existierend vorausgesetzten Lösungen der Gleichung $z = F(x, y)$ sind (Quasigruppeneigenschaften). Nach einer Charakterisierung der Funktionen G und nach verschiedenen Interpretationen und Anwendungen in der Nomographie und auf die Bisymmetrie-, Assoziativitäts- und ähnliche Gleichungen geht Verf. zu Funktionalgleichungen mit mehreren unbekannten Funktionen über.

Er löst die verallgemeinerten Assoziativitäts-, Transitivitäts- und Bisymmetriegleichungen

$$F(G(x, y), z) = H(x, K(y, z)),$$

$$F(x, y) = G(H(x, z), K(y, z)),$$

$$F(G(x, y), H(u, v)) = I(J(x, u), K(y, v)),$$

wo je eine Gleichung 4-4 bzw. 6 unbekannte Funktionen bestimmt, unter Stetigkeits- und Quasigruppenvoraussetzungen für reelle Funktionen und erhält im wesentlichen dieselben Lösungen wie M. Hosszú [Studia Math. **14** (1953), 100–106; Acta Sci. Math. Szeged **15** (1954), 203–208; Publ. Math. Debrecen **3** (1954), 205–214; MR **15**, 962; **16**, 371; **17**, 236] unter Derivierbarkeitsvoraussetzungen.—Seit der Abfassung und vor Erscheinen der vorliegenden Arbeit, aber von dieser nicht unabhängig erhielten J. Aczél, V. D. Belouszov, M. Hosszú, G. Pickert und F. Radó [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 51–56; MR **21** #4198; pp. 55–56; Acta Math. Acad. Sci. Hungar. **11** (1960), 127–136; und Mathematica-Cluj, im Erscheinen] noch allgemeinere Ergebnisse auf abstrakten Quasigruppen.

Nach weiteren Anwendungen auf spezielle Funktionalgleichungen, Gewebe und Nomogramme endet die Arbeit mit Verallgemeinerungen auf Funktionen von mehr als zwei Veränderlichen.—Die Veröffentlichung dieses wertvollen zusammenfassenden Artikels in einer internationalen Kongresssprache wäre sehr wünschbar.

J. Aczél (Debrecen)

6711:

★Epstein, L. Ivan. Nomography. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1958. x+134 pp. $4.50.

Of the eight chapters into which the book is divided, only the subject matter of chapters VII and VIII is not encountered in the current text books on nomography, and presents an interest to a mathematically well-trained reader interested in the theory underlying nomography. But for such a reader the 12 pages on determinants and 5 on matrices are superfluous. It would however be of scarce practical value even to such a reader. No canonic forms of nomograms are given. The few types given are not sufficiently explained. No mention is made of cases in which moduli have to be introduced. All the nomograms are derived from determinants; no geometric constructions are given.

D. Mazkewitsch (Cincinnati, Ohio)

6712:

Bauer, F. L.; und Stetter, H. J. Zur numerischen Fourier-Transformation. Numer. Math. **1** (1959), 208–220.

For $f(t)$ defined in $(-\infty, \infty)$ a "support function" $\hat{f}(t)$ relative to a given set of abscissas $t_\mu = t_0 + \mu h$ ($0 \le t_0 < h$, $\mu = 0, \pm 1, \pm 2, \cdots$) is defined by $\hat{f}(t) = h \sum_\mu f(t_\mu)\delta(t - t_\mu)$ with $\delta(t)$ the Dirac function. The formal Fourier transform $\hat{F}(\nu)$ of $\hat{f}(t)$ gives a known interpolation expression for the Fourier transform $F(\nu)$ of $f(t)$. "Relations are established which exist between the spectrum $F(\nu)$ of a function $f(t)$ and the spectrum $\hat{F}(\nu)$ of its support function $\hat{f}(t)$. It is examined, then, how for certain classes of functions the numerically inaccessible spectrum F can be obtained from the spectrum $\hat{F}$." Analogous considerations apply to Fourier coefficients. A few examples are given to illustrate the method. The paper is related to previous work by W. Quade and L. Collatz [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1938**, 383–429] and W. Dällenbach [Arch. Elektrotech. **10** (1921), 277–282].

W. Gautschi (Oak Ridge, Tenn.)

6713:

Rabinowitz, Philip; and Weiss, George. Tables of abscissas and weights for numerical evaluation of integrals of the form $\int_0^\infty e^{-x}x^n f(x)dx$. Math. Tables Aids Comput. **13** (1959), 285–294.

The table of constants for Gaussian type quadrature formulas with weight function $e^{-x}x^n$ on $[0, \infty]$ are tabulated by the authors. It is stated that the tables are good to 18 significant figures for $n = 0(1)5$. The number of evaluation points k for $n = 0$ has range 5(4)33 and for $n > 0$, $k = 5(4)17$.

P. C. Hammer (Madison, Wis.)

COMPUTING MACHINES

See also 6677, 6678, 7073, 7116, 7134.

6714:

Gill, S. Current theory and practice of automatic programming. Comput. J. **2** (1959), 110–114.

This paper contains a general description of the present state of "automatic programming" (use of computing machines to aid in the pre-computational portion of problem solving, as well as in the actual solution process itself). It is a good background paper for someone unfamiliar with automatic programming systems to read, although it overlooks much pertinent work both in the United States and the Soviet Union. To United States computer-oriented mathematicians who are interested in the present state of this art in Great Britain, the author's discussion presents a fair picture of present British activity and opinion. *J. W. Carr, III* (Chapel Hill, N.C.)

6715:

★Boscher, Jean. Résolution par analogie électrique d'équations aux dérivées partielles du quatrième ordre intervenant dans divers problèmes d'élasticité. Préface de J. Pérès. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 348, 1958. xxi+130 pp. 2251 francs.

In this monograph, the author summarizes the findings and experience gained at the Ministère de l'Air in finding approximations to the solutions of some fourth order partial differential equations by the use of electrical analogies. The problems considered include the static deflections of plates and beams and the normal modes of these same devices when undergoing vibrations. The document exploits the use of a technique which has been given the name of "superposed networks". In effect, the fourth order equation is broken into two second order equations in two unknown functions. The simulation is then made by two electrical networks, each being in the form of a grid with resistive connections between junction points, and finally resistors connected between the two networks and between each network and external sources. It is worth noting that the arguments in favor of this approach start with the fourth order equation and then invoke a series of mathematical manipulations plus approximations to arrive at the nature of the analog.

The monograph gives an excellent summary of our present position with respect to analogies constructed in this fashion, their potential accuracy, and their physical realization. The realization of boundary conditions is discussed in great detail.

This reviewer is often unable to understand why persons who wish to construct electrical analogies start with a differential equation and then try by a succession of manipulations to break it down into a set of simpler items which may perhaps stand in correspondence with some sort of an electrical system. If in fact one starts with the physical system which gave rise to the differential equations in the first place, then the construction of the analogy is generally straightforward. Take the example of the elastic beam which the author treats. The basic physical facts are that a small section of the beam executes linear motions which are describable by a second order equation, the section also undergoes rotary motions also describable by a second order equation, and the two sorts of motions are coupled by a workless constraint. If then one simulates each of these factors just as they stand, there is no trouble in constructing the analog nor is there any great difficulty in satisfying boundary conditions. Unfortunately, all too few scientists seem to be conscious of the fact that the fourth order equations mentioned above are simply eliminations effected between two second order systems. *H. M. Trent* (Washington, D.C.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 6311, 6452, 6463, 6804, 7077.

6716:

★Герц, Г. Принципы механики, изложенные в новой связи. [Gerc, G. [Hertz, H.]. **The principles of mechanics, explained in a new connection.**] Translated by V. F. Kotov and A. V. Sulimo-Samuĭlo. Institut istorii estestvoznaniya i tehniki. Klassiki nauki. Izdat. Akad. Nauk SSSR, Moscow, 1959. 386 pp. (1 plate) 15 rubles.

In addition to the text, this book contains Russian translations of the essays on Hertz by Helmholtz and Poincaré. There is also a special essay, "The principal mechanical ideas of Henry Hertz", by A. T. Grigoryan and L. S. Polak, containing a brief biography, "The philosophical bases of Hertz's mechanics", "Hertz's book ... and its place in the development of mechanics", "The concept of 'force' in the mechanics of Hertz". There are also some annotations, a list of Hertz's works, and a bibliography of works about Hertz, many of them in Russian. *C. Truesdell* (Bloomington, Ind.)

6717:

★Froda, Alexandre. La finitude en mécanique classique, ses axiomes et leurs implications. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 238–249. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

The author emphasizes the necessity of distinguishing between a movement μ defined by a kinematical equation $\bar{r}=\bar{r}(t)$ only and a realizable movement μ_c. Every μ_c is, within the framework of the present paper, supposed to satisfy a system (R) of axioms in classical mechanics. But the converse is not necessarily true.

The main problem discussed is how to establish a system (R) not containing the axiom R_3 presupposing the existence of an acceleration. It is the author's contention that this can be achieved by altering the formulation of some axioms and by introducing the two following new "axioms de finitude". If a movement μ is realizable during a finite interval of time δ, it satisfies: (F1) Among the sequences of regular, realizable movements having μ as their limit, there exists a sequence $\mu_n=1, 2, \cdots$, where the forces $\bar{F}_n$ are "bornees dans leur ensemble"; (F2) the force $\bar{F}(t)$ in the movement μ cannot change direction an infinity of times in the course of an infinite sequence of instants within δ.

These axioms apply to classical mechanics only, but the author does not exclude the possibility of their extension to wave and relativistic mechanics.

E. B. Schieldrop (Oslo)

6718:

★Артоболевский, И. И.; Левитский, Н. И.; и Черкудинов, С. А. Синтез плоских механизмов. [Artobolevskiĭ, I. I.; Levitskiĭ, N. I.; and Čerkudinov, S. A. Synthesis of plane mechanisms.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 1084 pp. 34.90 rubles.

Eine wirklich gute Erweiterung, mit grosser Sachkenntnis geschrieben, der im Jahre 1944 in Moskau von Artobolevskiĭ, Bloh und Dobrovol'skiĭ veröffentlichten *Sintez mehanizmov* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1944; MR **13**, 788], die nun auch die von den Verfassern und andern Forschern inzwischen erzielten neuen Forschungsergebnisse umfasst.

Das reichhaltige Material ist—ausser einer Einführung, in der die historische Entwicklung der Getriebesynthese dargestellt ist—in drei Teile unterteilt.

Im ersten Teil (Kapitel I und II) werden kurz Fragen des Getriebeaufbaus und der kinematischen Geometrie der Getriebe behandelt.

Im zweiten Teil (Kapitel III–VIII) sind die exakten Methoden der Getriebesynthese dargestellt. Es werden Probleme der Synthese der Zentroidalgetriebe mit drei Elementen, der Getriebe mit vier und fünf Elementen und zentroiden Roll- und Wälzelementenpaaren untersucht. Ferner wird die Synthese der Getriebe mit niederen Elementenpaaren behandelt. Weiter ist die Synthese einer Reihe von Getrieben, die Kegelschnitte beschreiben und solcher, die Kurven höherer Ordnung beschreiben, sowie von Getrieben, die gewisse funktionale Abhängigkeiten wiedergeben, hervorzuheben. Die Synthese der Kurvengetriebescheiben und der sogenannten Maltheserkreuze bilden den Gegenstand anderer Kapitel.

Die Theorie der Näherungsmethoden für die Getriebesynthese ist im 3. Teil des Buches ausgeführt (Kapitel IX–XVI), mit einem besonders verdienstvollen und aufschlussreichen Kapitel "Die Synthese der Übertragungsgetriebe" (Kapitel IX). Es bringt Zusammenfassungen und wichtige Ergebnisse, die den verschiedenen Problemen der Tschebyscheffschen Minimal abweichung einer gegebenen Funktion gewidmet sind. In diesen Kapiteln ist ferner die Synthese der approximativen Führungsgetriebe sowie die Synthese der Übertragungsgetriebe mittels Interpolationsmethode und Methode der besten Näherung sowie die Synthese der Rastgetriebe behandelt.

Es werden geometrische und algebraische Methoden angewandt. Reichhaltige Literaturangaben (die Bibliographie am Ende des Buches ist 10 Seiten stark) und eine Reihe von Beispielen mit direkter Anwendungsmöglichkeit in der Praxis erhöhen die Nützlichkeit des Buches für den Fachmann. Das Buch enthält ebenfalls ein ausführliches Namen- und Sachregister. Es bildet so einen wertvollen Beitrag zu der auf diesem Spezialgebiet vorhandenen Literatur. Es kann zugleich als Lehrbuch und besonders als ein vortreffliches Nachschlagewerk für Entwurfsingenieure, Hochschullehrer, Forscher und Lernende, die mit den Grundlagen der Getriebelehre vertraut sind, dienen. Die Übersetzung dieses trefflichen und tadellos gestalteten Buches in mehrere andere Sprachen ist sehr empfehlenswert.

D. Mangeron und R. Bogdan (Jassy und Bucharest)

6719:

★Артоболевский И. И. Теория механизмов для воспроизведения плоских кривых. [Artobolevskiĭ, I. I. Theory of mechanisms for reproduction of plane curves.] Problems in the Theory of Machines. Akademiya Nauk SSSR, Institut Mašinovedeniya. Izdat. Akad. Nauk SSSR, Moscow, 1959. 255 pp. 13.80 rubles.

This book, the latest in the series of monographs, Problems on Theory of Machines, contains a very detailed study and results of the author's sustained scientific activity in the field [Results 2nd All-Union Conf. Fundamental Problems in Theory of Machines and Mechanisms, Moscow, March 24–28, 1958, Izdat. Akad. Nauk SSSR, Moscow, 1958; Proc. 3rd Congr. Theoret. Appl. Mech., Bangalore, Dec. 24–27, 1957, pp. 293–300, Indian Soc. Theoret. Appl. Mech., Indian Inst. Tech., Kharagpur, 1958; MR **21** #986]. It also extends the related results obtained in U.S.S.R. and in other countries.

Proceeding from the most recent investigations in the theory of mechanisms which describe given paths, the author emphasizes the part played by such mechanisms in the construction of various machines and automatic apparatuses through the obtainment of different functional dependences. The main novelty in the present treatment is the systematic use of both the analytical and geometrical methods.

Other problems discussed are: Theory of mechanisms with higher pairs of IV class and that of mechanisms with lower pairs of V class describing various curves and the theory and synthesis of exact rectilinear and circular guiding mechanisms.

Further, the mechanisms describing ellipses, hyperbolas, parabolas and cissoids, strophoids, cardioids, conchoids are dealt with.

The theory and synthesis of mechanisms reproducing algebraic curves of the IVth order and the theory of the mechanical reproducing of a class of curves higher than IVth order are the subjects of another chapter. Also included are problems pertaining to the synthesis of mechanisms reproducing usual transcendent curves as well as usual curves of high order of parabolic and hyperbolic type.

Some open questions (partly with conjectured solutions) are formulated.

The last paragraph (§ 33) develops a highly interesting historical account concerning new and old researches in the field. This very readable and attractively printed book is a valuable contribution to the theory of mechanisms for the reproduction of plane curves. It solves some of a series of problems posed by the construction of automatic machines. Undoubtedly, it will be translated, like the author's other book, into other languages.

D. Mangeron and R. Bogdan (Jassy and Bucharest)

6720:

Volmer, J. Die Sonderfälle der Burmesterschen Mittelpunktkurve mit Doppelpunkt und ihre getriebetechnische Bedeutung. Rev. Méc. Appl. **4** (1959), no. 2, 301–322.

This is a translation from the Romanian [Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 425–448; MR **21** #6112].

O. Bottema (Delft)

6721:

Volmer, J. Konstruktion der Doppelpunkte einer Koppelkurve der Schubkurbel. Z. Angew. Math. Mech. **39** (1959), 328–331.

6722:

Cassini, Ugo. Su un teorema di Peano ed il moto del polo. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 631–655.

Consider the motion of a material system T consisting of a rigid part C and a residual part M (rigid or not), the latter being in an arbitrary motion with respect to C [see G. Peano, Atti Accad. Torino **30** (1895), 515–523, 845–852; Atti Accad. Naz. Lincei. Rend. Cl. Sci. Mat. Nat. (5) **5** (1896), 163–168].

The author gives the formulas for the velocity and position vectors of an arbitrary point of the rigid part C in the form of finite and linear expressions. Peano's theorem on the velocity of an arbitrary point of C when no external forces are acting on the system follows as a particular case of these formulas.

Applications made by Peano to the motion of the center of inertia of a system, which is free of any external forces and the barycentric ellipsoid of inertia of which is an ellipsoid of revolution, and to the motion of the terrestrial pole are confirmed anew.

Finally numerical calculations are given to evaluate the influence of the Gulf stream on the motion of the terrestrial pole. The results obtained are in agreement with those of Peano who did not give the actual calculations.

E. Leimanis (Vancouver, B.C.)

6723:

Egorov, V. A. On Bonnet's theorem. J. Appl. Math. Mech. **22** (1958), 1025–1036 (721–729 Prikl. Mat. Meh.).

If a certain number of mass-points m_i starting from the same position A with codirectional initial velocities $\mathbf{v}_{i0}$ under the action of forces $\mathbf{F}_i$, functions of the position only, describe the same curve (AB), then according to Bonnet's theorem a mass-point M starting at A with velocity $\mathbf{V}_0$ codirectional with the $\mathbf{v}_{i0}$ will be taken along the same curve by a force $\mathbf{F}$ equal to the resultant of the $\mathbf{F}_i$ if in A the kinetic energy of M equals the sum of kinetic energies of the m_i.

According to the generalised theorem of Egorov the same holds true with a force

$$\mathbf{F} = a_1\mathbf{F}_1 + a_2\mathbf{F}_2 + \cdots + a_n\mathbf{F}_n \tag{1}$$

if and only if

$$MV_0^2 = a_1m_1v_{10}^2 + a_2m_2v_{20}^2 + \cdots + a_nm_nv_{n0}^2 > 0 \tag{2}$$

and, inversely, if the same curve (AB) is described by M under the action of $\mathbf{F}$ in (1), then the first part of (2) follows.

The author applies his theorem to the problem of a particle gravitationally attracted by two fixed masses and to the particular solutions with elliptic and hyperbolic orbits obtained by Lagrange and Legendre. He shows that Bonnet's theorem cannot be applied to the hyperbolic solution as supposed by Bonnet himself and later authors. By the aid of Egorov's theorem, which because of the arbitrary constants a_i in (1) admits the introduction of repulsive forces ($a_i < 0$), the problem of existence regions and other properties of hyperbolic motion can be fully solved.

E. B. Schieldrop (Oslo)

6724:

Bogoiavlenskii, A. A. On some special solutions of the problem of motion of a heavy rigid body about a fixed point. J. Appl. Math. Mech. **22** (1958), 1049–1064 (738–749 Prikl. Mat. Meh.).

Continuing an earlier investigation of the motion when the center of gravity of the body is located on one of the principal axes of inertia, the author gives a form of the differential equations and indicates a method for solving them. Then, with one constant restricted so as to yield an integrable case, the details are worked out to the point where identification can be made with previously known solutions, including those of Steklov, Goriachev, Kowalewski, and Chaplygin.

P. Franklin (Cambridge, Mass.)

6725:

Aczel, Otto. Umkehrung des Problems der Brachistochrone. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 207–210 (1959). (Romanian. German and Russian summaries)

Das Problem der Brachistochrone besteht darin, dass eine durch zwei Punkte gehende Kurve gesucht wird, so dass ein Massenpunkt unter Einwirkung eines Kräftefeldes den Weg zwischen beiden Punkten, entlang dieser Kurve in kürzester Zeit durchquert. Hier wird das Problem umgedreht: es wird eine Kurve gegeben und man sucht jenes Kräftefeld, für welches die Kurve eine Brachistochrone ist. Man findet ein System zweier partieller Differentialgleichungen, deren Lösung eben das Potential des Feldes darstellt. Zum Schluss wird ein Beispiel angeführt.

Zusammenfassung des Autors

6726:

Quilghini, Demore. Sul principio dell'effetto giroscopico. Ricerche Mat. **7** (1958), 205–231.

Consider a solid S with a fixed point O, and assume that the trihedral $T(x, y, z)$ with the origin at O and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along its axes is rigidly connected with S and represents the principal inertia trihedral of S with respect to O. Denote by A, B, C the principal moments of inertia of S with respect to the x, y, z axes respectively and by $\boldsymbol{\omega}\ (p, q, r)$ the angular velocity of S with respect to T. Further assume that the solid S is subject to a large initial rotation $\boldsymbol{\omega}_0\ (p_0, q_0, r_0)$ about the z-axis (the latter being an axis of stable rotation and $p_0, q_0, 1/r_0$ being infinitesimal quantities of the same order) and to external forces the moment $\mathbf{M}$ of which with respect to O is orthogonal to $\mathbf{k}$, i.e., $M_z = 0$. Under the additional assumption that $A + B \neq C$, Stoppelli [Giorn. Mat. Battaglini (4) **4** (**80**) (1951), 14–38; MR **12**, 760] established to within a vector, the modulus of which is of the order $1/r_0$, the equation of motion of S in the form

$$Cr_0\frac{d\mathbf{k}}{dt} = \mathbf{M} + \frac{A-B}{A+B-C}\mathbf{M}_1 + \mathbf{M}_2, \tag{1}$$

where $\mathbf{M}_1 = M_x\mathbf{i} - M_y\mathbf{j}$ is the symmetric vector of $\mathbf{M}$ with respect to the x-axis and $\mathbf{M}_2$ is a vector the modulus of

which is finite and which vanishes only for $p_0=q_0=0$ and $\mathbf{M}=0$ at $t=0$.

In a second paper, Stoppelli [Rend. Sem. Mat. Univ. Padova **21** (1952), 25–43; MR **14**, 421] assumed that $M_z\neq 0$ but did not generalize formula (1) for this case.

The present author is concerned with the planar case of S ($A+B=C$ and $A\neq B$) when the procedures developed in the above two papers break down. Assuming that $|\dot{\mathbf{M}}|$ remains bounded as $r_0\to\infty$ it is shown that for $A+B\neq C$ the vector $\dot{\mathbf{k}}$ to within a vector, the modulus of which is of the order $1/r_0^2$, satisfies the equation

$$Cr_0\frac{d\mathbf{k}}{dt}=\mathbf{M}-M_z\mathbf{k}+\frac{1}{2}\frac{A-B}{A+B-C}\mathbf{M}_1+\frac{1}{2}\frac{B-A}{A+B-C}\mathbf{M}_2+\mathbf{M}_3.$$

Here the vectors $\mathbf{M}_1=M_x\mathbf{i}-M_y\mathbf{j}+M_z\mathbf{k}$ and $\mathbf{M}_2=-M_x\mathbf{i}+M_y\mathbf{j}+M_z\mathbf{k}$ are symmetric to $\mathbf{M}$ with respect to the planes $y=0$ and $x=0$ respectively, and $\mathbf{M}_3$ is a finite vector which vanishes only for $p_0=q_0=0$ and $\mathbf{M}\times\mathbf{k}=0$ at $t=0$.

Passing to the planar case of S ($A+B=C$), it is shown that under the above assumption on $|\dot{\mathbf{M}}|$ the modulus of $\dot{\mathbf{k}}$ is of the order of the modulus of $(A-B)\mathbf{k}\times\mathbf{I}$, where $\mathbf{I}=\int_0^t \mathbf{M}d\tau$. The first of these vectors vanishes for $A=B$ but in general it remains finite as $r_0\to\infty$. Therefore, for a planar case and a non-gyroscopic structure of S ($A\neq B$), the gyroscopic phenomenon of the tenacity of the axis of large rotation will not be evident even if S rotates rapidly about an axis orthogonal to the plane containing the mass of S.

The expressions obtained directly for p and q under the assumption that S is planar are also obtainable from the corresponding expressions of p and q for the case $A+B\neq C$ when passing to the limit as $A+B\to C$.

E. Leimanis (Vancouver, B.C.)

6727:

Rionero, Salvatore. Sul principio dell'effetto giroscopico nel caso sferico. Ricerche Mat. **7** (1958), 281–290.

Denote by θ, ψ, φ the Eulerian angles of T with respect to a fixed trihedral T', and exclude from consideration the spherical and planar cases of S when $A=B=C$ and $A+B=C$ respectively. Assuming that $\mathbf{M}$ is of the form (1) $\mathbf{M}=\mathbf{k}\times\mathbf{F}$, where $\mathbf{F}$ depends on t, r, θ, ψ but is independent of p, q and the angle φ of proper rotation of S, Stoppelli showed in the first of his papers that the equation (2) $Cr_0d\mathbf{k}/dt=\mathbf{M}$ determines to within infinitesimal quantities of higher order in $1/r_0$ the Eulerian angles and hence the position of S.

In the second paper mentioned in the review above, Stoppelli considered the case when the external forces acting on S are reducible to two vectors, one of which is applied at O and the other is $(Q, \mathbf{F})$ with the moment $\mathbf{M}$ with respect to O. Let Q' be the orthogonal projection of Q on the z-axis and $\mathbf{M}^*$ the moment of $(Q', \mathbf{F})$ with respect to O. Then the Eulerian angles are determined to within terms of higher order in $1/r_0$ by equation (2*) which is obtained from (2) by replacing r_0 by $\bar{r}_0=r_0+[(1/Cr)\times(\partial M_z/\partial\varphi)]_{t=0}$ and $\mathbf{M}$ by $\mathbf{M}^*$.

The present paper is concerned with the discussion of the question as to what extent the above results of Stoppelli hold or can be generalized for the spherical case. It is shown that if $\mathbf{M}$ is of the form (1), equation (2) holds also for the spherical case of S if $\mathbf{F}$ depends on p and q as well as on t, r, θ, ψ. However, the dependence of $\mathbf{F}$ on φ may destroy the validity of (2) [S. Rionero, Ricerche Mat. **7** (1958), 14–20; MR **20** #4366]. Furthermore, it is shown that equation (2*) cannot approximate the position of S, even under the assumption made by Stoppelli that Q and $\mathbf{F}$ are independent of p, q, φ.

E. Leimanis (Vancouver, B.C.)

6728:

Chirikov, B. V. The passage of a nonlinear oscillating system through resonance. Dokl. Akad. Nauk SSSR **125** (1959), 1015–1018 (Russian); translated as Soviet Physics. Dokl. **4**, 390–394.

6729:

Kharlamova, E. I. A particular case of the integrability of the Euler-Poisson equations. Dokl. Akad. Nauk SSSR **125** (1959), 996–997 (Russian); translated as Soviet Physics. Dokl. **4**, 313–315.

The motion considered is that of a rigid body containing a cavity filled with a homogeneous liquid, moving about a fixed point. The body starts with the liquid at rest. The author imposes two conditions which determine uniquely the direction of the initial vector from the fixed point to the center of gravity. He also requires two inequalities to be satisfied by the principal moments of inertia. For this special case he shows that the equations of the title can be integrated in terms of elementary functions.

P. Franklin (Cambridge, Mass.)

6730:

Rosenberg, R. M.; and Atkinson, C. P. On the natural modes and their stability in nonlinear two-degree-of-freedom systems. J. Appl. Mech. **26** (1959), 377–385.

In a non-linear symmetrical two-degree-of-freedom system one of the natural modes is in-phase ($x_2=x_1$) and the other is out-of-phase ($x_2=-x_1$); for a system with cubic non-linear terms the equations for the in-phase and out-of-phase modes, defined by x_i and x_o respectively, are

$$\ddot{x}_i+a_1x_i+a_3x_i^3=0,$$
$$\ddot{x}_o+(a_1+2A_1)x_o+(a_3+8A_3)x_o^3=0.$$

The stability of these modes depends on that of solutions of a Mathieu equation and is investigated in detail. In particular, if there is a single value of frequency and amplitude at which the system can vibrate in either mode (for this amplitude the coupling spring characteristic has zero slope), the out-of-phase mode experiences a change of stability.

G. B. Warburton (Edinburgh)

6731:

Proskuriakov, A. P. On ways of introducing a small parameter into equations of nonlinear vibrations. J. Appl. Math. Mech. **22** (1958), 1005–1009 (711–713 Prikl. Mat. Meh.).

This paper is a cursory discussion of the effect on the problem of determining periodic solutions caused by different ways of introducing a small parameter. The reason for publishing the paper must be to provide simple examples. For the most part the changes observed can be seen to be merely a change in a coefficient.

J. P. LaSalle (Baltimore, Md.)

6732:
Lyah, R. A. On the influence of atmospheric resistance upon the motion of the artificial satellite. Byull. Inst. Teoret. Astr. **7**, 321–326 (1959). (Russian. English summary)

6733:
Gel'fgat, B. E. Über zwei Integrierbarkeitsfälle des Zweikörperproblems bei veränderlichen Massen und deren Anwendung zur Untersuchung der Bewegung in einem widerstehenden Mittel. Byull. Inst. Teoret. Astr. **7**, 354–362 (1959). (Russian. German summary)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 6636, 6920, 6958, 6969, 7037, 7039.

6734:
Vedenov, A. A. A new method in classical statistical physics. Dokl. Akad. Nauk SSSR **125** (1959), 757–760 (Russian); translated as Soviet Physics. Dokl. **4**, 364–368.

This paper is concerned with a method for the evaluation of the partition function in statistical mechanics by expansion in powers of the inverse temperature and Fourier transformation of the two-particle correlation function. It is shown that the method gives the usual result for the second virial coefficient of a gas with radially symmetric intermolecular forces. The most interesting application, however, is to a completely ionized gas, where the Debye-Hueckel formulae come out as a first approximation, and high temperature corrections can be computed.

H. S. Green (Adelaide)

6735:
★Chisholm, J. S. R.; and de Borde, A. H. An introduction to statistical mechanics. International Series of Monographs on Physics, Vol. 2. Pergamon Press, New York - London - Paris - Los Angeles, 1958. ix + 160 pp. $6.00.

This textbook is a simple presentation of a few elements and applications of statistical mechanics, suitable for a short course at the senior or first year graduate level primarily for physics students. The concept of ensembles is introduced at once and the microcanonical ensemble is discussed in detail with an elementary version of the method of steepest descent by Darwin and Fowler. (The pedagogic value of this choice is debatable.) Applications include the ideal gas, ortho-para hydrogen, fluctuation, and elementary discussion of the electron theory of metals, ferromagnetism, the Debye crystal, and equilibrium phenomena.

J. Ross (Providence, R.I.)

6736:
Katsura, Shigetoshi. Fourth virial coefficient for the square well potential. Phys. Rev. (2) **115** (1959), 1417–1426.

A calculation is given of the fourth virial coefficient D for molecules interacting with a square-well potential. Difficult multiple integrals are evaluated by means of a Fourier transformation and the addition theorem of Bessel functions. The temperature variation of D is considered and a general discussion is given.

S. Simons (London)

6737:
Beckmann, Petr. The probability distribution of the velocities of gas molecules after collision with a rapidly moving rigid body. Czechoslovak J. Phys. **9** (1959), 557–561. (Russian summary)

Un corps rigide est en mouvement uniforme (vitesse V) au sein d'un gaz de particules ayant une distribution des vitesses maxwellienne (vitesse d'agitation thermique u). La probabilité d'une vitesse $v=|\mathbf{V}+\mathbf{u}|$ est

$$p(v) = \frac{v}{V\sigma}\exp\left(-\frac{V^2+v^2}{2\sigma^2}\right)\left(\frac{2}{\pi}\right)^{1/2}\sinh\frac{vV}{\sigma^2}, \quad \sigma=\left(\frac{kT}{m}\right)^{1/2}.$$

La valeur moyenne de v et sa variance sont calculées. Ces résultats peuvent être appliqués au calcul de la probabilité d'ionisation d'une particule du gaz par collision contre le corps rigide.

J. Naze (Marseille)

6738:
Huang, Kerson. Energy levels of a Bose-Einstein system of particles with attractive interactions. Phys. Rev. (2) **115** (1959), 765–777.

"An N-body Bose-Einstein system of particles with long-range attraction and hard-sphere repulsion between particles is considered. It is shown that if the constants of the interaction have values within a certain range it is possible to calculate the ground-state energy of the system as a function of Ω/N, where Ω is the volume of the box containing the system, in the limit $N\to\infty$, $\Omega\to\infty$, with $N/\Omega=\rho$ fixed. The results show that the system can possess an N-body bound state, which has an equilibrium density and negative energy, and that the interactions can be saturating. Excited states are also considered. It is shown that low-lying excitations consist purely of phonons, whose velocity agrees with that computed from the macroscopic compressibility, furnished by the ground-state energy. The formula for the general excited energy levels suggests that thermodynamically the system may have a 'gas' phase and two 'liquid' phases, the transition between the two 'liquid' phases being the analog of the Bose-Einstein condensation of the ideal gas. Thermodynamic considerations are, however, not contained in this paper." (From the author's summary)

R. M. Evan-Iwanowski (Syracuse, N.Y.)

6739:
Domb, C.; Maradudin, A. A.; Montroll, E. W.; and Weiss, G. H. Vibration frequency spectra of disordered lattices. I, II. Phys. Rev. (2) **115** (1959), 18–36.

These two papers describe an extensive study of the normal modes for a linear chain of particles with nearest neighbor harmonic forces wherein the mass of any particle is to be m with probability τ and $M\neq m$ with probability $1-\tau$ independent of the mass of other particles (disordered chain). In part I, exact expressions are derived for the $2n$th moments of the frequency spectrum, first in a combinatorial form and then in terms of generating functions. These are evaluated explicitly for even moments through the twentieth. In part II two different methods are used to derive an approximate spectrum density from moments through the twentieth. This is evaluated with a computing machine for a range of concentrations and ratios of the two masses. Graphs of the approximate spectrum density show the transition in behavior from that of a monatomic chain of mass m to one of mass M as

the relative concentration changes from zero to one, and they also indicate the presence of impurity bands for low concentration of light atoms. In addition to this work, which is mainly numerical, the authors also give an exact analysis of the spectrum when one of the masses is infinite. The infinite mass effectively cuts the chain into finite segments with rigid walls and the spectrum becomes a highly irregular function defined on rational points. Near the maximum frequency, however, an approximate density exists and it vanishes exponentially fast as the frequency approaches its maximum value. The behavior of this spectrum is quite peculiar. *G. Newell* (Providence, R.I.)

6740:

Strunin, B. M. On the statistical theory of strain in metals. Dokl. Akad. Nauk SSSR **125** (1959), 790–793 (Russian); translated as Soviet Physics. Dokl. **4**, 439–442.

The strain in metals is considered on a statistical basis starting from the observation that an actual piece of metal is composed of a very large number of individual crystallites. Since these individual small crystals, in general, will differ in composition, orientation, size, etc., a description of the deformation process can only be made by resorting to statistical techniques. A normal distribution is assumed for the elastic energy in the various microcrystals. Equations are derived for the hardening curve and for the conditions of rupture.

H. Statz (Waltham, Mass.)

6741:

Tauber, G. E. Variational principle for transport phenomena. J. Franklin Inst. **268** (1959), 175–221.

A general treatment is given of the variational principle in transport theory, and its application to the calculation of transport coefficients. The theory covers inter-band scattering, an external magnetic field and lattice vibrations. *S. Simons* (London)

6742:

Rice, Stuart A.; and Kirkwood, John G. On an approximate theory of transport in dense media. J. Chem. Phys. **31** (1959), 901–908.

A new treatment is given of transport in dense media. Three principal approximations are employed: (a) the expansion of the gradient of the pair interaction potential between molecules at time $t+s$ about the gradient at time t, and the neglect of all terms higher than the second, (b) the use of a local equilibrium distribution function in pair space, and (c) the approximation of the pair diffusion tensor as the direct sum of singlet diffusion tensors. The validity of these approximations is discussed. Various physical quantities are evaluated theoretically, and the results compared with experiment for liquid argon. *S. Simons* (London)

6743:

Smets, Henri B. Exact solution of the reactor kinetics equations for linear, exponential, or reciprocal reactivity dependence on time. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 256–271. (French summary)

The reactor kinetics equations (r.k.e.) are considered on the basis of the one-velocity-group model. When this model is used, all the reactor characteristics capable of changing enter into the r.k.e. only via a single parameter, the reactivity. If the reactivity changes linearly with time, so do the coefficients in the r.k.e. Then these ordinary differential equations are of the type known to be solvable by means of the Laplace transform method. If the reactivity changes exponentially with time or as $k_0 - 1/\gamma t$, the Laplace transform method turns out to be also applicable.

The Laplace transform inversion is considered in detail, and in the case of linear variation of reactivity the solution is expressed as a series in confluent hypergeometric functions, while in the case of exponential variation it is expressed in terms of a finite number of Meijer's G-functions. In the case of one group of delayed neutrons the above solutions are reduced to closed expressions in terms of a finite number of confluent hypergeometric functions. The asymptotic forms of the solution for the cases of very slow and very rapid changes in reactivity are also considered. *B. Davison* (Toronto, Ont.)

6744:

Rice, Stuart A. Statistical mechanical theory of transport processes. XIII. Kinetic theory of dense rigid sphere fluids. J. Chem. Phys. **31** (1959), 584–594.

"This paper discusses the details of the calculation of the coefficients of shear viscosity and thermal conductivity for a moderately dense fluid of rigid spheres based on a modified Boltzmann equation previously derived. The physical significance of short time smoothing and its effect on the computed transport coefficients is briefly considered." (Author's abstract)

H. L. Frisch (Murray Hill, N.J.)

6745:

Gombás, P.; und Kisdi, D. Zur statistischen Theorie des Nukleonengases bei beliebigen Temperaturen. Z. Physik **156** (1959), 125–130.

"Es wird die statistische Theorie eines Nukleonengases für beliebige Temperaturen entwickelt und das Grundgleichungssystem des Problems hergeleitet. Für die Wechselwirkung der Nukleonen wird eine Linearkombination von Wignerschen, Majoranaschen, Heisenbergschen und Bartlettschen Kräften angesetzt."

Zusammenfassung der Autoren

6746:

Philippot, J. L'irréversibilité dans l'espace des variables angulaires. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 591–594. (English summary)

6747:

Coleman, Bernard D.; and Noll, Walter. Conditions for equilibrium at negative absolute temperatures. Phys. Rev. (2) **115** (1959), 262–265.

6748:

Buchdahl, H. A. Über Approximationen der Zustandsgleichung eines Fermi-Dirac-Gases. Ann. Physik (7) **3** (1959), 345–351.

The author obtains simple algebraic approximations for the pressure and specific heat (at constant volume per

particle) of an ideal Fermi-Dirac gas in any state of degeneration. Starting from known series [see, e.g., E. C. Stoner, Philos. Mag. (7) **21** (1936), 145–160] these approximations are developed in a mainly heuristic way. In conclusion, numerical results from the new formulas are compared with known table-values.

W. Rheinboldt (Syracuse, N.Y.)

6749:

Weiss, George. On the inversion of the specific-heat function. Progr. Theoret. Phys. **22** (1959), 526–530.

The paper presents a method for the inversion of experimental data on the specific heat C_v of a solid to find its vibrational frequency spectrum. The resulting formula is an alternative to that developed by Grayson-Smith and Stanley [J. Chem. Phys. **18** (1950), 236]. Neither of the formulae is directly applicable to experimental results because the series contains coefficients which are derivatives of $C_v(T)$ and, therefore, only one or two of them can be determined with sufficient accuracy.

J. Kestin (Providence, R.I.)

6750:

Fradkin, E. S. The Green's functions method in quantum statistics. Ž. Èksper. Teoret. Fiz. **36** (1959), 1286–1298 (Russian); translated as Soviet Physics. JETP **9**, 912–919.

The paper deals with the formulation of quantum-statistical mechanics and many-body problems in terms of Green's functions. It is claimed that all previous formulations, such as the Debye-Huckel, Hartree-Fock, Thomas-Fermi, and GellMann-Bruckner methods are contained in the resultant equations. Special attention is devoted to obtaining the time-dependent Green's function. The example of particles interaction according to Coulomb's law is discussed in detail, with two limiting cases: (1) Uniform distribution of ions; (2) a correlation of ions and electrons so that an appreciable violation of translational invariance takes place. An application to superconductivity is promised in a forthcoming paper.

M. J. Moravcsik (Livermore, Calif.)

6751:

Fradkin, E. S. Functional equation and operator solution for the partition function. Dokl. Akad. Nauk SSSR **125** (1959), 66–68 (Russian); translated as Soviet Physics. Dokl. **4**, 327–330.

Field-theoretical methods, such as time-ordered operators, are used to obtain an expression for the partition function of a canonical ensemble. The Hamiltonian includes the interaction of a single fermion field with a scalar boson field with zero rest mass. {The translation is poor and sometimes wrong: e.g., the sentence after equation (16) should start "it is easy" and not "it is difficult".}

D. ter Haar (Oxford)

6752:

Fradkin, E. S. The method of Green's functions in quantum statistics. Dokl. Akad. Nauk SSSR **125** (1959), 311–314 (Russian); translated as Soviet Physics. Dokl. **4**, 347–350.

The system of functional equations derived in an earlier paper [see review above] is solved using Green's function methods.

D. ter Haar (Oxford)

6753:

Rastogi, R. P.; and Srivastava, R. C. Generalized theory of thermal transpiration and thermal diffusion based on the thermodynamics of irreversible processes. Physica **25** (1959), 391–397.

6754:

Fáy, Gy.; und Tábori, Gabrielle. Diskussion des zeitlichen Verlaufes von thermodynamischen Prozessen. Acta Phys. Acad. Sci. Hungar. **10** (1959), 129–133. (Russian summary)

6755:

Magalinskiĭ, V. B. On the evaluation of coordinate probabilities for nonlinear systems by Gibbs' method. Ž. Èksper. Teoret. Fiz. **36** (1959), 1423–1427 (Russian); translated as Soviet Physics. JETP **9**, 1011–1014.

Let the state of a system be characterized by the generalized co-ordinate Q and generalized velocity $\dot{Q}$; the external force acting on Q by $F(Q)$. We are interested in the probability distribution $W(Q, \dot{Q}; Q_0, \dot{Q}_0)$ which tells us the probability of finding the system which is a member of an equilibrium ensemble, with a value Q for its co-ordinate at time t, if this co-ordinate had the value Q_0 at $t = t_0$. The author constructs the Fokker-Planck equation associated with this problem utilizing two observations: (1) the Fourier transform of W, denoted by Z, is formally related to the partition function of an auxiliary ensemble in which an additional constant force $-a$ is acting in the direction of Q; hence the averages of $\dot{Q}$ and Q^n over this auxiliary ensemble can be expressed in terms of Z and its derivatives; (2) the averages of $\dot{Q}$ and Q^n over this ensemble have to satisfy the averaged equation of motion for a system in the presence of an additional force $-a$; because of (1) the averaged equation of motion is then a relation between the Fourier transform of W and its derivatives. In terms of W itself, this relation turns out to be the usual Fokker-Planck equation of the problem.

This way the author constructs the Fokker-Planck equation for a ensemble of electrical circuits in which the current-voltage characteristics are non-linear, the current being a power series in the voltage. The author solves formally the resulting equation and investigates the solution in the low temperature limit. He finds that in this approximation the Brownian motion of the charge in the circuit is still described by a W which is Gaussian; however, in contrast to the linear case the dispersion depends not only on the time, but also on the initial conditions.

N. L. Balazs (Princeton, N.J.)

6756:

Résibois, P. Theorie formelle du scattering classique. Physica **25** (1959), 725–732.

It is pointed out that one can use the (two-particle) Liouville equation to develop a scattering formalism in classical mechanics which is fully analogous to the S-matrix technique of quantum mechanics.

D. ter Haar (Oxford)

ELASTICITY, PLASTICITY

See also 6740, 6836b, 6896a–c, 7063, 7085, 7086.

6757:

★Noll, Walter. The foundations of classical mechanics in the light of recent advances in continuum mechanics. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 266–281. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

The author attempts to clarify the foundations of classical continuum mechanics by axiomatizing the subject. The concept of a body, kinematics, forces and constitutive equations all receive some attention. He has performed real service by stating explicitly non-obvious assumptions tacitly made by others in deriving the existence of a stress tensor. In his footnote 4, there is the statement "... dependence of the stress vector ... on the curvature ... is impossible". Logically, all that can be concluded from the facts given is that such dependence is inconsistent with the assumptions introduced. The author's remark seems to imply that the conventional theory of surface tension is unsound.

J. L. Ericksen (Baltimore, Md.)

6758:

★Kondo, Kazuo. (Editor) RAAG memoirs of the unifying study of basic problems in engineering and physical sciences by means of geometry. Vol. II. Gakujutsu Bunken Fukyu-Kai, Tokyo, 1958. xvi+589 pp. (4 plates) $17.50; £6 5s.

This large volume is the second general report on the research activities of a group of Japanese applied physicists and engineers. It contains twenty-eight papers grouped under the following headings (in parenthesis we have indicated some of the topics considered by abbreviating some of the titles): A. Linear Geometry and Topology of Networks (Four terminal and AC Networks, Theory of trees and multitrees, duality of electric, magnetic and dielectric networks); B. Differential Geometry of Engineering Dynamical Systems (Study of rotor windings of non-Riemannian generalized machines, non-holonomic foundations of hydro-aerodynamical forces); C. Geometry of Deformations and Stresses (Elasticity and biharmonic tensors, the Weissenberg effect); D. Non-Holonomic Geometry of Plasticity and Yielding (Uncertainty elements in plasticity phenomena); E. Geometry of Observation and Structurology (Quaternionian structurology, fundamental principle of observation, Quantum-statistical study of the spectrum of turbulence); F. Contributions to Diakoptics (Codiakoptics, Generalized Diakoptics); G. Topological Information Theory of Engineering System Structures (Communication and transportation networks, oriented switching circuits); and Miscellaneous subjects (Aberrations of optical systems, interval algebra and its applications to numerical analysis, the number of trees appearing in certain modular structures). As may be seen, the papers cover a wide range from a somewhat philosophical discussion of "Generalized Diakoptics" to a set of tables on the number of trees in the molecular structure of hydrocarbons. Thus any general statements are not really applicable to most of the papers.

In general, however, the papers are concerned with applying modern mathematical techniques (primarily tensor analysis, geometry and combinatorial topology) to the problem of formulating a viewpoint from which various engineering questions can be considered and, possibly, understood. In this context, the attempt is of importance even if not always entirely successful (for example, the fibre bundle of observation is not well-defined and may not be appropriate as used; however fibre bundles may well be of value in some of the geometric situations considered) and many of the papers merit careful study. The mathematical techniques are more up-to-date than what seems to be usual in papers dealing with engineering questions and, for example, Eilenberg and Steenrod's *Foundations of algebraic topology* [Princeton Univ. Press, 1952; MR 14, 398] is quoted as often as Veblen's *Analysis situs* [Amer. Math. Soc., New York, 1922]. However, it is disturbing to see at times weird statements such as the following (p. 254): "In other words, the rational field is extended to the field of real numbers so that the latter's topological characteristics defined by the Haussdorf separation axiom may replace the order relation of rational numbers."

J. B. Giever (University Park, N.M.)

6759:

Ferrarese, Giorgio. Sulla relazione simbolica della meccanica dei sistemi continui vincolati. Rend. Mat. e Appl. (5) **17** (1958), 305–312.

This paper proves that a certain variational inequality is equivalent to Cauchy's statical equation and surface condition for an incompressible continuous medium subject to surface constraints.

C. Truesdell (Bloomington, Ind.)

6760:

Kil′čevs′kiĭ, M. O.; and Tkačuk, G. I. On certain properties of integral equations derived on the basis of the theorem of reciprocity of work. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 210–212. (Ukrainian. Russian summary)

6761:

Conway, H. D. Analysis of plane stress in polar coordinates and with varying thickness. J. Appl. Mech. **26** (1959), 437–439.

"The biharmonic stress-function equation for plane stress in polar coordinates is generalised by assuming that the thickness varies with the radius. In order to make this equation tractable the thickness is assumed to be proportional to an arbitrary power of the radius. Mitchell's general solution for the constant-thickness case is then generalised for this variation of thickness. As a numerical example, the solution is given for the problem of a segmental plate bent by end couples, the constant-thickness version of this well known problem being due to Golovin [Transactions of the Institute of Technology, St. Petersburg, 1891]." (Author's summary)

R. M. Morris (Cardiff)

6762:

Savruk, M. A. Concentration of stress in a thin isotropic strip weakened by two unequal circular apertures, in the

case of pure displacement. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 327–330. (Ukrainian. Russian summary)

6763:
Kroupa, František. Stress and deformation of an infinite strip, caused by an edge dislocation. Apl. Mat. **4** (1959), 239–254. (Czech. Russian and English summaries)

The solution is established for an edge-dislocation across a strip of infinite length by superposition of the known stress-function for the dislocation in infinite space with the stress-functions ensuring stress-free boundaries of the infinite strip without dislocation.

A. M. Freudenthal (New York, N.Y.)

6764:
Ufliand, Ia. S. Elastic equilibrium of an infinite body weakened by an external circular crack. J. Appl. Math. Mech. **23** (1959), 134–144 (101–108 Prikl. Mat. Meh.).

The author considers an infinite elastic body, cut over the surface $z=0$, $x^2+y^2>a^2$. Solutions for symmetric and anti-symmetric loading (with respect to z) are given in terms of Mehler-Fock integrals

$$\int_0^\infty f(\tau)P^m_{-\frac{1}{2}+i\tau}(x)d\tau.$$

Explicit results are given for the particular cases of isolated normal forces on the sides of the cut.

J. W. Craggs (Newcastle-upon-Tyne)

6765:
Tarabasov, N. D. Stress state of multiply connected half-plane loaded by discs. Inžen. Sb. **25** (1959), 136–144. (Russian)

6766:
Rvachev, V. L. The pressure on an elastic half-space of a stamp with a wedge-shaped planform. J. Appl. Math. Mech. **23** (1959), 229–233 (169–171 Prikl. Mat. Meh.).

The author uses a harmonic function of form $r^n\phi(\theta,\psi)$, where r, θ, ψ are spherical coordinates, to find the pressure on a surface of elastic half-space resulting from application of an infinite wedge-shaped stamp; the solution can be applied to find stresses in the neighborhood of corners of polygonal stamps. *C. E. Pearson* (Cambridge, Mass.)

6767:
Collins, W. D. Note on displacements in an infinite elastic solid bounded internally by a rigid spherical inclusion. J. London Math. Soc. **34** (1959), 345–351.

The author uses a harmonic vector function to modify prescribed displacements in an infinite body so as to correspond to an internal spherical rigid boundary; the solution is also applied to the limiting case of slow steady viscous flow of a fluid about a rigid sphere. Specialization is made to a problem considered by Goodier—that of a rigid spherical inclusion in an infinite body under tension.

C. E. Pearson (Cambridge, Mass.)

6768:
Derkačev, A. A. Some generalizations of the theory of V. Z. Vlasov for calculation of thin-walled rods. Inžen. Sb. **26** (1958), 273–279. (Russian)

6769:
Stavraki, L. M. Equations of elastic equilibrium of arbitrarily loaded thin rods. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 167–178. (Ukrainian. Russian and English summaries)

"The paper deals with the space problem of the equilibrium of a thin strained rod.

Several assumptions are made, and a mathematical statement of the problem is given, which leads to a system of four ordinary differential equations. It is shown that these equations may approximately be considered as linear differential equations with variable coefficients."

Author's summary

6770:
Gužovs'kiĭ, V. V. Eccentrically compressed thin-walled rods of open profile within and outside the limits of elasticity. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 179–190. (Ukrainian. Russian and English summaries)

6771:
Amenzade, Yu. A. Torsion of a prismatic beam with square cross-section reinforced by a circular rod. Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Teh. Him. Nauk **1958**, no. 2, 35–53. (Russian. Azerbaijani summary)

6772:
Stevens, L. K. Control of collapse mechanisms in triangulated frames. Austral. J. Appl. Sci. **10** (1959), 138–155.

The unfavourable characteristics of compression members impose severe limitations in the application of plastic theory to the analysis of triangulated frames. This paper discusses these limitations, and demonstrates that certain types of trusses can be designed either so that yielding occurs in tension only, or so that compression members are the last to yield. The work is presented elegantly.

J. Heyman (Cambridge, England)

6773:
Bassali, W. A. Torsion of beams whose sections are bounded by certain quartic curves. J. Mech. Phys. Solids **7** (1959), 272–281.

Cauchy integral methods are used to determine the torsion function for an isotropic cylinder whose cross section in the z-plane can be mapped on the unit circle $|\zeta|\leq 1$ by the transformation

$$z = c\zeta/(1+n\zeta+m\zeta^2), \quad c>0.$$

Such sections may be identified with the inverse of an ellipse with respect to a point on its major axis. For $n=0$, the section is Booth's lemniscate, while for $m>0$ and $n=\pm 2\sqrt{m}$ the sections become elliptic limaçons. Expressions for the torsional rigidity and shearing stresses are found in the general case and these reduce to those found for the Booth's lemniscate and elliptic limaçons by other

authors. The peripheral shear stress is computed and graphed for the two special cases $m=-0.2$, $n=-0.4$ and $m=0.2$, $n=-0.6$. *R. M. Morris* (Cardiff)

6774:

Abramian, B. L. Torsion of a circular cylindrical shaft with a conical part. J. Appl. Math. Mech. **22** (1958), 952–958 (679–683 Prikl. Mat. Meh.).

This paper deals with the determination of the torsion function for a shaft of variable cross-section consisting of a circular cylinder as one part and a circular truncated cone of angle 2α as the other part. The shaft is considered to be twisted under the action of couples applied over its end sections together with any applied arbitrary loading on the curved surface of the cylindrical part. The curved surface of the conical part is assumed free from loading.

R. M. Morris (Cardiff)

6775:

Rvachev, V. L. On bending of an infinite beam on elastic half-space. J. Appl. Math. Mech. **22** (1958), 984–988 (698–700 Prikl. Mat. Meh.).

Solutions of a three-dimensional contact problem of a strip have been utilized to solve the title problem when the base of the beam before deformation is plane and bending takes place in the longitudinal direction. Following Štaerman, generalized Hertz theory has been taken and the deflection has been obtained for an arbitrary load being in the form of Fourier integral. *S. C. Das* (Madras)

6776:

Green, A. E. The equilibrium of rods. Arch. Rational Mech. Anal. **3**, 417–421 (1959).

Starting with the three dimensional equations of equilibrium, the author defines stress and couple resultants appropriate for thin rods and, without approximation, derives equations of equilibrium which they satisfy. To the best of the reviewer's knowledge, previous derivations all involve unnecessary approximations.

To complete the theory, one must supplement these equations with suitable constitutive equations. The author includes brief comments concerning the latter, but gives no definite formulation. *J. L. Ericksen* (Baltimore, Md.)

6777:

Teodorescu, Petre P. On the strength calculation of thick plane plates. Rev. Méc. Appl. **4** (1959), no. 2, 323–333.

Formal doubly infinite solutions in terms of biharmonic functions are obtained for an infinite thick plate with faces subjected to periodic loads. No numerical results are obtained. *B. R. Seth* (Kharagpur)

6778:

Maksimenko, V. P. Calculation of a thick square plate clamped on three sides and loaded by a concentrated force. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 200–209. (Ukrainian. Russian and English summaries)

"The author considers a thick square plate clamped on three sides. The problem is solved in finite differences on the basis of a method of displacements in the spatial problem of the theory of elasticity." *Author's summary*

6779:

Schumann, Walter; et Martinola, Marzio. Étude théorique et expérimentale de la répartition des tensions tangentielles dans les plaques circulaires d'épaisseur variable. C. R. Acad. Sci. Paris **249** (1959), 1608–1610.

The basic assumptions of the classical (linear) theory of thin plates of uniform thickness imply that the stresses parallel to the middle surface $z=0$, i.e., σ_x, σ_y, τ_{xy}, are linear functions of z. As an immediate consequence of the equilibrium equations (neglecting body forces) one obtains the result that the shearing stresses τ_{xz}, τ_{yz} are quadratic in z. The authors essentially extend this result to certain plates of variable thickness $h(r)$, whose middle surfaces (of revolution) $z_0(r)$ have a slight initial curvature (r, z = cylindrical coordinates). Precisely, they establish the following fact. Consider small axisymmetric deformations of a circular curved isotropic plate of radius R and let the continuous functions $h(r)$, $z_0(r)$ satisfy the conditions $h \ll R$, $dh/dr = O(h/R)$, $dz_0/dr = O(h/R)$. If σ_z and the tangential displacements u, v are linear in z (Euler-Bernoulli-Kirchhoff hypothesis), then the stress τ_{rz} is quadratic in z. Experimental measurements are reported which confirm the formula obtained for τ_{rz}.

{Reviewer's note. It appears that the authors' result can be deduced without reference to the Euler-Bernoulli-Kirchhoff hypothesis simply by assuming σ_x, σ_y, τ_{xy} to be linear in z. The restriction to circular plates and axisymmetric deformations also seems unnecessary.}

H. J. Weinitschke (Los Angeles, Calif.)

6780:

Kączkowski, Zbigniew; and Żórawski, Marek. Tables for computation of influence surfaces for a plate strip. Rozprawy Inż. **7** (1959), 39–91. (Polish. Russian and English summaries)

Tables of influence surfaces have been computed for an elastic strip freely supported along the parallel edges which permit the evaluation of deflections and moments under the action of concentrated forces, concentrated couples or shear forces, as well as of uniformly distributed, circularly distributed and knife edge loads. Examples of such strips supported by rows of internal columns are evaluated. *A. M. Freudenthal* (New York, N.Y.)

6781:

Sherman, D. I. On the transverse bending of a plate supported along the edges and consisting of several closed curves. J. Appl. Math. Mech. **23** (1959), 145–164 (109–123 Prikl. Mat. Meh.).

The author considers the deflection of the middle surface of a plate occupying the finite, multiply-connected region S bounded by smooth closed curves L_j ($j=1, 2, \cdots, m+1$) of which L_{m+1} is the external boundary; the curve L_j encloses the region S_j. He assumes, for the complex potential functions $-i\phi(z)$ and $-i\int\psi(z)dz$, the following expressions.

$$\phi(z) = \phi^{(0)}(z) + \sum_{j=1}^{m} iA_j(z-z_j)\ln(z-z_j),$$

$$\psi(z) = \psi^{(0)}(z) - \sum_{j=1}^{m} [(iA_j\bar{z}_j + 2\bar{B}_j)\{\ln(z-z_j) + z_j/(z-z_j)\} + (1+i)D_j/(z-z_j)],$$

where z_j is any fixed point in S_j, and $\phi^{(0)}(z)$, $\psi^{(0)}(z)$ are analytic functions in S defined as follows:

$$\phi^{(0)}(z) = \int_L \{\omega(t)\theta_1(t) + \bar{\omega}(\bar{t})\theta_2(t)\}G(t, z)dt,$$

$$\psi^{(0)}(z) = \int_L \{\omega(t)H(t, z) + \bar{\omega}(\bar{t})T(t, z)\}dt \qquad (L = L_1 + L_2 + \cdots + L_{m+1}).$$

The functions $\theta_1(t)$, $\theta_2(t)$, $G(t, z)$, $H(t, z)$, $T(t, z)$ are known functions of t and z. The "density function" $\omega(t)$ involved in the above integrals is the function to be determined from the boundary condition

$$\lambda[\phi'(t) - \bar{\phi}'(\bar{t})] + \bar{t}\bar{\theta}_1(\bar{t})\gamma(t) - \dot{t}\theta_2(t)\bar{\gamma}(\bar{t}) + \Gamma\{\omega(t), t\} = f(t),$$

where λ depends on Poisson's ratio and the variable t is assumed to lie on the boundary L; dots denote differentiation with respect to the arc and $\gamma(t) = \phi(t) - t\bar{\phi}'(\bar{t}) - \bar{\psi}(\bar{t})$. The operator Γ depends on the deflection at certain points of the plate and $f(t)$ is a certain function determined by the nature of the loading. The coefficients A_j, B_j, D_j appearing in the above expressions are also given as contour integrals round the curves L_j of certain functions of t and z_j, involving the density function $\omega(t)$. Applying the above expressions for $\phi(z)$ and $\psi(z)$ in the above boundary condition and "after performing certain calculations requiring a series of transformations", the author obtains the following Fredholm integral equation for the density function $\omega(t)$:

$$\omega(t_0) + \int_L \{\omega(t)M(t, t_0) + \bar{\omega}(\bar{t})N(t, t_0)\}dt + O\{\omega(t), t_0\} = f(t_0),$$

where t_0 is a certain point on L and $M(t, t_0)$, $N(t, t_0)$ are given functions of t and t_0 and $O\{\omega(t), t_0\}$ is given in terms of $\omega(t)$, t and t_0. The author demonstrates the validity of the above expressions for $\phi(z)$ and $\psi(z)$ by establishing the uniqueness of solution of the above integral equation.

The author then considers the case where the region S extends to infinity and is bounded by the curves L_j $(j = 1, 2, \cdots, m)$. This problem is also solved on the same lines as the previous case; the modified expressions for $\phi(z)$ and $\psi(z)$ are assumed in terms of a density function $\omega(t)$, which was shown to satisfy a similar integral equation. *M. Nassif* (Assiut)

6782:
Alekseev, S. A. Calculation of a circular elastic membrane under uniform transverse loading. Inžen. Sb. **25** (1959), 64–80. (Russian)

6783:
Šeremet'ev, M. P. Bending of an infinite plate weakened by an elliptic hole whose edge is reinforced by a thin ring. Inžen. Sb. **25** (1959), 51–63. (Russian)

6784:
Lipkin, V. B. Stress concentration in an orthotropic plate weakened by a circular hole in the case of pure bending. Inžen. Sb. **26** (1958), 179–187. (Russian)

6785:
Bassali, W. A.; and Dawoud, R. H. Bending of a thin circular plate concentrically loaded under a general boundary condition. Proc. Math. Phys. Soc. Egypt. no. 21 (1957), 1–18 (1958). (Arabic summary)

"The complex variable method is applied to obtain solutions for the deflection of a supported circular plate with loadings $p_0 r^n \cos m\theta$ and $p_0 r^n \sin m\theta$, m, n being integers, first over the whole plate and then over a concentric circle under a general boundary condition including clamped and hinged boundaries. The deflection, stress couples and stress resultants at any point of the plate due to any load (over the whole plate or over a concentric circle), which admits of Taylor's expansion in r and Fourier's expansion in θ, can therefore be obtained by the principle of superposition." (Authors' abstract)

R. M. Morris (Cardiff)

6786:
Kurdin, N. S. Bending of a partially loaded rectangular plate with two holes and two free edges. Inžen. Sb. **25** (1959), 104–110. (Russian)

6787:
Zhgenti, V. S. A study of the equilibrium equations of thin elastic shells of positive Gaussian curvature. J. Appl. Math. Mech. **23** (1959), 179–194 (134–145 Prikl. Mat. Meh.).

6788:
Nagibin, L. N. On stresses in a heavy anisotropic half-plane weakened by two circular holes. Inžen. Sb. **25** (1959), 122–135. (Russian)

6789:
Tao, L. N. On toroidal shells. J. Math. Phys. **38** (1959/60), 130–134.

Thin toroidal shells under axisymmetrical loading are considered. Following the work of Clark [J. Math. Phys. **29** (1950), 146–178; MR **12**, 557] the governing equations are expressed as a single equation in the complex domain. This is now transformed to Heun's equation and solved in terms of Heun's functions. These functions are expressed in power series and are shown to be convergent for the whole shell. *L. S. D. Morley* (Farnborough)

6790:
Mesmer, G. Über eine Gruppe von Singularitäten im Membranspannungszustand der Kugelschale. Ing.-Arch. **28** (1959), 208–212.

Zunächst werden die möglichen Lastfälle erörtert, die eine Singularität im Membranspannungszustand erzeugen können, und dann die bekannten Differentialgleichungen für die Schnittkräfte mit den Lösungen angeben. Es werden neun verschiedene Fälle entsprechend den möglichen Singularitäten behandelt. Für die wichtigsten der in der Arbeit behandelten Fälle sind die Spannungstrajektorien auf der Kugeloberfläche unter der Wirkung der singulären Lasten skizziert. *W. Zerna* (Hannover)

6791:

★**International Union of Theoretical and Applied Mechanics. Non-homogeneity in elasticity and plasticity: Symposium, Warsaw, September 2–9, 1958.** Edited by W. Olszak. Pergamon Press, New York-London-Paris-Los Angeles, 1959. xxiv+529 pp. $15.00.

This book contains summaries of 52 papers presented at the Symposium on Non-Homogeneity in Elasticity and Plasticity held at Warsaw, September 2–9, 1958. Two survey papers discuss the present state of the subject and others give solutions to beam, plate, shell and three-dimensional problems; experimental results are also included, as are applications to physical metallurgy. The papers are grouped in 6 sections: 1. Elasticity, 2. Plasticity, 3. Rheology, 4. Dynamics and Wave Propagation, 5. Statistical and Micro-Non-Homogeneity, 6. Varia.

6792:

Ambarcumyan, S. A.; and Pešmaldžyan, D. V. The theory of orthotropic shells and plates. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **12** (1959), no. 1, 43–59. (Russian. Armenian summary)

The basic equations of shell theory are formulated under the assumption that the length of normals to the middle surface remains unchanged after loading, that the normal stresses in the direction of the thickness are essentially independent of the stress components parallel to the middle surface, and that the variation of transverse shears through the thickness may be arbitrarily prescribed. It is also assumed that the principal axes of the orthotropic elasticity tensor are either parallel or normal to the middle surface at every point.

For the case of shells of positive Gaussian curvature with rectangular edge layout, the system of fundamental equations is reduced to one tenth order partial differential equation in a combined "stress-displacement" function. Finished results are given for the cases of a spherical and a cylindrical shell and a flat plate under sinusoidal loading and with freely supported edges. Numerical comparisons are made with Kirchhoff's theory.

Similarly, the analysis is carried out in polar coordinates for circular plates under uniform load and with different degrees of isotropy and conditions of edge support.

F. T. Geyling (Summit, N.J.)

6793:

Signorini, Antonio. Estensione delle formule di Almansi a sistemi elastici anisotropi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 246–253.

Per un solido elastico isotropo il potenziale elastico può sempre pensarsi come una funzione dei soli allungamenti principali, mentre per i solidi anisotropi questo potenziale non può più dipendere solo dagli allungamenti, ma i secondi membri delle espressioni delle tensioni principali dati da Almansi [Atti Accad. Lincei Rend. (5) **20** (1911), 705–714] conservano un semplice significato. Nella presente nota è dimostrato che, sempre in assenza di ogni vincolo interno, i secondi membri delle formule per le tensioni continuano a dare esattamente gli sforzi normali sugli elementi di superficie corrispondenti a quelli che nello stato di riferimento del solido appartengono ai piani principali di deformazione. Anche un'altra estensione delle formule di Almansi è indicata in questa nota, specificando un'equazione secolare di 3° grado avente per radici proprie le tre tensioni principali, mentre nel caso di isotropia essa si spezza in tre equazioni di 1° grado equivalenti alle formule di Almansi. *D. P. Rašković* (Belgrade)

6794:

Sobolevskiĭ, V. M. Elastic strained state of an anisotropic revolving cylindrical pipe acted upon by centrifugal force, internal and external pressure, axial force and radial flow of heat. Dokl. Akad. Nauk BSSR **2** (1958), 91–99. (Russian)

6795:

Kosmodamianskiĭ, A. S. Estimate of the accuracy of the principle of Saint-Venant in extension of an anisotropic strip. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 9, 130–133. (Russian)

The accuracy of calculating the extension (longitudinal and transverse) of a rectangular anisotropic strip by means of St. Venant's principle is estimated. This author applies to that purpose a mixed variational method and finds that the errors are much greater in longitudinal extensions than is the case in transverse extensions of such strips. The author states that a simple generalisation of the methods of double series, applied to the same problem for an isotropic strip by S. Timoshenko in his *Theory of elasticity* [McGraw-Hill, New York-London, 1934] and M. M. Filon'enko-Borodić [Akad. Nauk SSSR. Prikl. Mat. Meh. **10** (1946), 193–208; MR **7**, 437], is here of no use because the series obtained converge too slowly.

T. P. Anđelić (Belgrade)

6796:

Golecki, Józef. On the foundations of the theory of elasticity of plane incompressible non-homogeneous bodies. Arch. Mech. Stos. **11** (1959), 383–398. (Polish and Russian summaries)

Using the displacement potential F and the stress function Φ, the author investigates plane states of strain in non-homogeneous incompressible elastic bodies, which satisfy the equation

$$\gamma\nabla^4\varphi + 2\gamma_{,x}(\nabla^2\varphi)_{,x} + 2\gamma_{,y}(\nabla^2\varphi)_{,y} + 4\gamma_{,xy}\varphi_{,xy} + (\gamma_{,xx}-\gamma_{,yy})(\varphi_{,xx}-\varphi_{,yy}) = 0,$$

where $\varphi = F$ if $\gamma = \mu$ and $\varphi = \Phi$ if $\gamma = 1/\mu$ (μ = Lamé coefficient). The case of a thick-walled tube under inner and outer pressure is solved in detail, for $\mu = Cr^2$ and $\mu = C/r^2$ (C = constant, r = radius). It is claimed that for non-homogeneous bodies one can be led to a pseudo-solution with multiple-valued displacements.

J. Nowinski (Madison, Wis.)

6797:

Rutecki, Jerzy. Vibration of space frames of thin-walled elements with open cross-section. Rozprawy Inż. **7** (1959), 93–142. (Polish. Russian and English summaries)

The modes of vibration of an elastic bar with thin-walled cross-section, symmetric with respect to two perpendicular axes, are investigated, and the results applied to the investigation of the vibrations of simple rectangular two- and three-dimensional frames made up of such bars, under

a number of simplifying assumptions concerning interaction at the corners.

A. M. Freudenthal (New York, N.Y.)

6798:

Babuška, Ivo. Die nichtlineare Theorie der inneren Reibung. Apl. Mat. **4** (1959), 303–321. (Russian. Czech and German summaries)

In this report the non-linear theory of mechanical steady vibrations, taking into account the existence of a hysteresis loop which is a function of amplitude only and is entirely independent of the vibration frequency, has been proposed.

From the physical point of view, this theory is a development of that proposed by N. N. Davidenkov. Mathematically, this hypothesis leads to an equation relating the strain $\varepsilon(t)$ to the corresponding stress $\sigma(t)$:

$$(1) \qquad \dot{\sigma}(t) = f_1(\sigma, \varepsilon)\cdot\dot{\varepsilon}_+(t) + f_2(\sigma, \varepsilon)\cdot\dot{\varepsilon}_-(t),$$

where $\dot{\sigma}(t)=d\sigma(t)/dt$, $\dot{\varepsilon}(t)=d\varepsilon(t)/dt$, $\dot{\varepsilon}_+(t)=\max[0, \dot{\varepsilon}(t)]$, $\dot{\varepsilon}_-(t)=\min[0, \dot{\varepsilon}(t)]$. The mechanical properties of the material are defined by two functions $f_1(\sigma, \varepsilon)$ and $f_2(\sigma, \varepsilon)$. The author defines the properties of these functions and explains their physical significance. The non-linear equation (1) is thoroughly discussed. The existence and uniqueness of its solution and its regular dependence on initial conditions, as well as its stability, have been proved.

In this theory, the damping of harmonic vibrations is independent of their frequency, but is strongly influenced by their amplitude. The theory gives full explanation of the unsteady motion for free vibrations and predicts the variation of the hysteresis loop, which tends asymptotically to its "stationary" shape, for periodic vibration.

As an example, the author proposes two forms of functions f_1 and f_2, namely:

$$\text{(I)}\quad \begin{aligned} &f_1(\sigma, \varepsilon) = C_1\cdot\exp(\alpha_1\varepsilon)\cdot(D_1-\sigma)^{\beta_1} \quad \text{for } \sigma \leqq D_1,\\ &f_2(\sigma, \varepsilon) = C_2\cdot\exp(\alpha_2\varepsilon)\cdot(D_2+\sigma)^{\beta_2} \quad \text{for } \sigma \geqq -D_2,\\ &f_1 = 0 \text{ for } \sigma \geqq D_1,\quad f_2 = 0 \text{ for } \sigma \leqq -D_2; \end{aligned}$$

$$\text{(II)}\quad f_1 = C_1\cdot g_1(\varepsilon)\cdot h_1(\sigma),\quad f_2 = C_2\cdot g_2(\varepsilon)\cdot h_2(\sigma);$$

where
$$\begin{aligned} g_1(\varepsilon) &= (1+\alpha_1\varepsilon) && \text{for } \varepsilon \geqq -1/\alpha_1,\\ &= 0 && \text{for } \varepsilon \leqq -1/\alpha_1;\\ g_2(\varepsilon) &= (1-\alpha_2\varepsilon) && \text{for } \varepsilon \leqq 1/\alpha_2,\\ &= 0 && \text{for } \varepsilon \geqq 1/\alpha_2;\\ h_1(\sigma) &= (1-\beta_1\sigma) && \text{for } \sigma \leqq 1/\beta_1,\\ &= 0 && \text{for } \sigma \geqq 1/\beta_1;\\ h_2{}^2(\sigma) &= (1+\beta_2\sigma) && \text{for } \sigma \geqq -1/\beta_2,\\ &= 0 && \text{for } \sigma \leqq -1/\beta_2. \end{aligned}$$

C_i, D_i, β_i, α_i $(i=1, 2)$ are positive material constants. The first of these two examples has been extensively developed and discussed.

In the numerical example which follows, the functions of type I (in metric units) have been numerically defined:

$$f_1 = 2000\cdot e^{0.1\varepsilon}\cdot(4000-\sigma)^{0.12},$$
$$f_2 = 2000\cdot e^{-0.1\varepsilon}\cdot(4000+\sigma)^{0.12}.$$

The hysteresis loops for different steady amplitudes have been demonstrated, as well as the "approach" of the loop to "steady" shape, for every cycle of an initially unsteady periodic vibration that tends asymptotically to a "steady" limit which is a harmonic vibration.

It seems to the reviewer that this theory gives a satisfactory similarity with the experimental shape of hysteresis loops in metals.

The author intends to generalize in future papers his theory for many other related phenomena, such as decreasing free vibrations, some mechanical systems with one degree of freedom, magnetic hysteresis, etc., with appropriate empirical characteristic functions f_1 and f_2.

J. S. Nałęszkiewicz (Warsaw)

6799:

Kondrat'ev, A. S. Oscillatory properties of the equilibrium forms of a beam column. J. Appl. Math. Mech. **22** (1958), 754–757 (541–543 Prikl. Mat. Meh.).

The author deals with a column of variable rigidity subjected to axial compressive loads which, however, may vary along the column. By a consideration of the appropriate homogeneous integral equation he shows that for all common end conditions the equilibrium configurations of the column satisfy the classical Sturm-Liouville oscillation theorems.

W. E. Boyce (Troy, N.Y.)

6800:

Kalandiya, A. I. On a direct method of solution of an equation in wing theory with an application to the theory of elasticity. Mat. Sb. N.S. **42** (**84**) (1957), 249–272. (Russian)

Betrachtet wird die Prandtlsche Integro-Differentialgleichung der Tragflügeltheorie unter der Voraussetzung, daß die Verteilung von Flügeltiefe und Anstellwinkel längs der tragenden Linie einer Hölderbedingung genügen. An den Enden werden die üblichen Annahmen gemacht. Gewissen Problemen der Elastizitätstheorie zu Liebe, bei denen diese Gleichung auch eine Rolle spielt, wird weiter angenommen, daß die "Flügeltiefe" durchweg positiv oder durchweg negativ ist. Verf. zeigt unter Heranziehung einiger Sätze der Approximationstheorie, daß die Folge der beim Multhoppschen Näherungsverfahren sich ergebenden Approximationspolynome gleichmäßig gegen eine Lösung der Integro-Differentialgleichung konvergiert. {Der Referent erinnert daran, daß die Konvergenz des Multhopp-Verfahrens unter etwas stärkeren Voraussetzungen 1950 von Weissinger [Math. Nachr. **3** (1950), 316–326; MR **12**, 541] nachgewiesen wurde.}

K. Maruhn (Giessen)

6801:

Stepanov, R. D. On the flutter of cylindrical panels moving in a gas. Inžen. Sb. **25** (1959), 92–100. (Russian)

6802:

Livanov, K. K. Stability of a clamped panel in a hypersonic flow. Inžen. Sb. **25** (1959), 101–103. (Russian)

6803:

Richardson, A. S., Jr. Theoretical and experimental investigation of random gust loads. II. Theoretical formulation of atmospheric gust response problem. NACA Tech. Note no. 3879 (1957), 50 pp.

[For part I, see Hakkinen and Richardson, NACA Tech. Note no. 3878 (1957); MR **19**, 489.] "Equations of motion are derived for the dynamic response of an aircraft

to random atmospheric gust loads. These equations include the degrees of freedom of plunging, pitching, rolling, and an arbitrary number of elastic normal modes. Solutions of these equations are expressed in terms of a number of so-called primitive solutions obtainable by introducing the Dirac delta function. The solutions for center-of-gravity acceleration response and wing-root bending-moment response depend upon certain autocorrelations and cross correlations which enter the analysis. Results for simplified cases show that unsteady aerodynamic theory is not important for increasingly large values of the turbulence scale compared with values of the wing chord. However, the pitching degree of freedom exhibits an important effect as the turbulence scale increases. The results are also compared with the results of the usual sharp-edged-gust formula." (Author's summary)

K. Maruhn (Giessen)

6804:

Hacker, T. On the longitudinal stability of an aircraft under repeated disturbances. Rev. Méc. Appl. **4** (1959), no. 2, 229–236.

[A Romanian version appears in Acad. R. P. Romîne Stud. Cerc. Mec. Apl. **10** (1959), 69–76; MR **21** #3173.] It is proposed to determine upper bounds on repeated disturbances that will result in perturbations of the aircraft's motion not exceeding prescribed values. The familiar linearized equations of longitudinal motion are used; thus the parameters describing the aircraft are the usual "derivatives" of longitudinal stability analyses. The mathematical treatment follows I. G. Malkin [*Teoriya ustoičivosti dviženiya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **15**, 873]. The case treated here is disturbance of the aircraft by gusts and, in particular, vertical gusts. As a simplification, the variation of flying speed is neglected; thus the motion involved is due to the short-period oscillation. The result of the calculation is a formula for an upper bound on the maximum value of the vertical-gust velocity as function of the permissible perturbations (in incidence and angular velocity) and the stability derivatives. The author points out that the rotary (damping) derivatives are most important to minimize the response to these gusts.

W. R. Sears (Ithaca, N.Y.)

6805:

Crisp, John D. C. The equation of energy balance for fluttering systems with some applications in the supersonic regime. J. Aero/Space Sci. **26** (1959), 703–716.

The equation of energy balance relates the total energy of the system to that of the impressed aerodynamic forces, and determines, in essence, those amplitude and phase relations which are necessary for the existence of flutter. This paper shows how the equation can be interpreted (in a geometrical manner) to provide a criterion of stability (i.e. of decrease of system total energy) of practical value.

For a simple binary system, the radius of the "energy circle" indicates whether the flutter is violent or mild (or non-occurrent, if the circle vanishes). Higher order systems are also considered, and certain general theorems deduced. Several examples of supersonic flutter of binary systems are considered, and the theory is also applied to panel flutter.

A. W. Babister (Glasgow)

6806:

Bankova'ka, N. V. Stability of a rectangular orthotropic plate compressed by a step-load. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 336–340. (Ukrainian. Russian summary)

6807:

Brush, D. O.; and Field, F. A. Buckling of a cylindrical shell under a circumferential band load. J. Aero/Space Sci. **26** (1959), 825–830.

A classical stability analysis (assuming perfect elasticity, no defects and infinitesimal displacements) is given for buckling of a long thin cylinder under circumferential pressure distributed over a finite band. Reasonable exponential displacement functions are assumed, involving six coefficients which are determined from energy considerations. Good checks are obtained with previous theories for all-over pressure and circumferential line load.

An interesting feature is the inclusion of the case when the pressure is concentrated on the outer ridges of the deformation, as it would be if applied by a semi-rigid ring. The increase in total load required, compared to that for uniform pressure, is found to increase with the band width.

In the expressions for membrane strains, the authors use $(v+\partial w/\partial\phi)$ where others, including this reviewer, have used $\partial w/\partial\phi$ alone. This "refinement" is probably unimportant in such applications, which is very fortunate since it is unquestionably incorrect; according to it, a rigid-body rotation of the cylinder about its axis would produce enormous circumferential strains!

L. H. Donnell (Chicago, Ill.)

6808:

Keller, Herbert B.; and Reiss, Edward L. Spherical cap snapping. J. Aero/Space Sci. **26** (1959), 643–652.

The problem of axisymmetric deformations of a clamped shallow spherical shell under uniform external pressure is attacked by finite difference methods, employing an iterative procedure similar to the method of over-relaxation in solving systems of linear equations. The convergence of this process is supported by heuristic arguments and numerical evidence. (The authors have previously shown that for the special case of a flat plate the process does indeed converge for a limited range of the load parameter and a particular choice of the relaxation parameter [Comm. Pure Appl. Math. **11** (1958), 273–292; MR **20** #4961].)

Displacements and stresses are obtained for unbuckled and buckled equilibrium states and are displayed in graphical form. On the basis of the load deflection curves the authors give estimates for the (classical) buckling load which practically coincide with earlier results of the reviewer [J. Soc. Indust. Appl. Math. **6** (1958), 209–232; MR **20** #4962]. They also present numerical evidence for the possibility of snapping according to an energy criterion proposed by Friedrichs. Their calculations cover only the range of a shell geometry parameter for which the normal deflection is of a simple "mode", i.e., monotone decreasing from the apex towards the edge of the shell. In this range, their method seems applicable with practically no restriction on the load parameter.

H. J. Weinitschke (Los Angeles, Calif.)

6809:
Savin, G. M. Development of investigations of the dynamics of an inelastic thread of variable length. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **5** (1959), 136–141. (Ukrainian)

6810:
Gilbert, Freeman; and Knopoff, Leon. Scattering of impulsive elastic waves by a rigid cylinder. J. Acoust. Soc. Amer. **31** (1959), 1169–1175.

The diffraction of elastic waves by a rigid cylinder is treated with the aid of methods known from the analogous problems for electromagnetic waves. The high-frequency approximation is derived first; its mode expansion depends on Airy functions connected with Bessel functions the order and argument of which are large and nearly equal. The application of an inverse Laplace transform then leads to a corresponding approximation for the pulse solution (produced by a point source at $t=0$) for small values of t; this latter solution depends on another Airy function the argument of which accounts for the retardation due to the finite time of propagation. The "geometrical optics" approximations in the illuminated zones are derived with the aid of the saddlepoint method, as usual.

A characteristic feature of this problem is the generation by diffraction of both compression and shear waves even if the source emits only one type of these waves. A compression wave travelling along the cylinder as a surface wave, e.g., may produce a shear wave by "critical refraction". The propagation direction (away from the surface) of this wave depends on the ratio of the propagation velocities of the two mentioned waves, in a way similar to Snell's law. All geometric-optical properties are thus complicated by the simultaneous presence of compression and shear waves. *H. Bremmer* (Eindhoven)

6811:
Knopoff, Leon. Scattering of shear waves by spherical obstacles. Geophysics **24** (1959), 209–219.

The problem of the scattering of a plane S-wave by a perfectly rigid, infinitely dense sphere of radius a is set up and solved for a variety of values of $X=2\pi a/\lambda_s$, and for a Poisson ratio $\frac{1}{4}$ of the medium. Expansions valid for $X<1$ are used; the scattered wave is decomposed into radial, meridional and azimuthal displacement fields, whose phase and angular distributions are plotted separately. *F. Villars* (Cambridge, Mass.)

6812:
Lockett, F. J. Longitudinal elastic waves in cylinders and tubes—including thermoelastic effects. Proc. Edinburgh Math. Soc. **11** (1958/59), 153–159.

A two-point boundary value problem for an ordinary differential equation on a finite interval is equivalent to the problem with the same differential equation on the infinite interval, zero boundary conditions at infinity, and suitable discontinuities at the original end points. This fact is applied to systems of partial differential equations which are reduced to ordinary differential equations by integral transforms. *H. F. Weinberger* (College Park, Md.)

6813:
Lockett, F. J. A method of deriving solutions of a class of boundary value problems. Proc. Edinburgh Math. Soc. **11** (1958/59), 147–152.

A method presented by the author in the preceding paper is applied to the problem mentioned in the title. *H. F. Weinberger* (College Park, Md.)

6814:
Gerasimov, I. S. On a dynamic similarity problem for a conical shell. Dokl. Akad. Nauk SSSR **125** (1959), 989–991 (Russian); translated as Soviet Physics. Dokl. 4, 306–308.

A circular conical elastic shell under the action of a moving load (constant axially symmetric pressure normal to the surface of the cone) which propagates at a constant velocity v from the apex of the cone in the direction of the axis of symmetry is investigated. The thickness of the shell is assumed to be a linear function of the distance from the apex.

6815:
Antsiferov, V. S. Automodel problem of the penetration of a solid body into the ground. J. Appl. Math. Mech. **22** (1958), 1231–1239 (856–860 Prikl. Mat. Meh.).

The paper presents a similarity treatment for the penetration at zero angle of attack of a circular cone into the ground (method applicable for underground explosions). The similarity medium is a non-viscous ideal barotropic gas. The translation is not satisfactory in spots and hence it may be very desirable to have the original document on hand. *M. G. Scherberg* (Dayton, Ohio)

6816:
★Шестопалов, Л. М. Деформирование металлов и волны пластичности в них. [Šestopalov, L. M. Deformation of metals and plastic waves in them.] Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1958. 268 pp. 13 rubles.

This book is concerned primarily with the experimental and theoretical study of the dependence of the mechanical behaviour of metals upon effects due to rate-of-strain and temperature. Part I (pp. 7–76) is entitled 'review of basic papers on the determination of the mechanical characteristics of metals under shock loading'. Part II (pp. 77–263) is entitled 'experimental study of the laws of propagation of plastic strain for various types of rapid loading'.

The author first surveys previous work in the field. Attention is then given to a discussion of the experimental techniques for producing and for measuring plastic deformations in metals, more particularly when these occur under impact and so-called uni-axial stress conditions (e.g., as with Hopkinson pressure-bar apparatus). Experimental results obtained by the author are given and are discussed in relation to his theoretical analysis of time- and temperature-dependent effects in the mechanical behaviour of metals. Under the main heading 'experimental study of the laws of propagation of plastic strain', the chapter headings are 'shock loading by compression of a rod with a free end', 'shock loading by compression of a rod with a fixed end', 'shock concentrated loading on the end face of a long plate', 'shock concentrated loading on the end face of a short plate', and 'investigation of the

time dependence of stress in the dynamic loading of bodies'. Under the main heading 'discussion of results obtained', the chapter headings are 'equation of strain of an elastic-plastic medium', 'general analysis of the strain equation of an elastic-plastic medium', 'creep of metals', 'relaxation stress in metals', 'equation of motion of an elastic-plastic medium', 'problem of axial shock loading by compression of an elastic-plastic rod', and 'on the time dependence of stress of an elastic-plastic medium for dynamic loading'. From a theoretical viewpoint, the analysis given in this last set of chapters constitutes the most interesting part of the book. Unfortunately, in the absence of a full translation, it is not possible for the present reviewer to give adequate appraisal of this analysis which in part at least is based upon concepts in crystal physics. In spite of the title of the book, only very restricted cases of plastic waves (mainly the one-dimensional case) are considered and from this particular point of view the book is to be considered less general than the book by N. Cristescu [*Probleme dinamice in teoria plasticităţii*, Acad. R. P. Romîne, Şti. Tehn. 13, Bucharest, 1958].

H. G. Hopkins (Sevenoaks)

6817:

May, W. D.; Morris, E. L.; and Atack, D. Rolling friction of a hard cylinder over a viscoelastic material. J. Appl. Phys. **30** (1959), 1713–1724.

Using the approach due to D. Tabor [Philos. Mag. (7) **43** (1952), 1055–1059], the authors investigate the frictional resistance encountered by a rigid circular cylinder rolling over a viscoelastic material. This friction is regarded as being due largely to energy losses during deformation of the latter and thus arises as a bulk property of the deforming material; the frictional force is evaluated by considering the moment of the surface stresses about the axis of the rolling cylinder. The solutions for various kinds of linear viscoelastic material are examined. The determination of the relaxation time distribution from rolling friction data is also discussed.

J. E. Adkins (Nottingham)

6818:

Thurston, George B. Theory of oscillation of a viscoelastic medium between parallel planes. J. Appl. Phys. **30** (1959), 1855–1860.

The sinusoidal rectilinear oscillation of an incompressible viscoelastic medium between fixed infinite parallel planes is examined assuming a general linear relation between stress and strain and their time rates of variation. Calculations are given for the acoustic impedance of a given section of the planes using various assumptions, and the possibility of extending the results to rectangular tubes is briefly discussed. *J. E. Adkins* (Nottingham)

6819:

Walsh, J. B.; and Mackenzie, A. C. Elastic-plastic torsion of a circumferentially notched bar. J. Mech. Phys. Solids **7** (1959), 247–257.

6820:

Ivlev, D. D. Certain particular solutions of equations for the axisymmetrical problem in the theory of ideal plasticity and generalization of Prandtl's solution of compression of plastic layer between two rough plates. J. Appl. Math. Mech. **22** (1958), 943–951 (673–678 Prikl. Mat. Meh.).

A strictly inverse method is used in which the defining equations for the title problem are written out, particular simple functions are chosen to satisfy these equations, and the resulting boundary value problem is then determined. Using Mises' yield condition, the author considers a generalization of Prandtl's cycloidal solution in plane strain and shows that it solves the problem of pressing of a plastic cylindrical layer by rough coaxial cylindrical surfaces. Some inverse solutions are also given for Tresca's yield condition but no applications are indicated.

P. G. Hodge, Jr. (Chicago, Ill.)

6821:

Hodge, P. G., Jr.; and Sankaranarayanan, R. The determination of safe loads of beams subjected to combined twisting and biaxial bending moments. J. Appl. Mech. **26** (1959), 442–447.

A lower bound is established to the yield-point relation between the torque and bending moment (about an arbitrary axis) of a rigid/plastic square section beam with Tresca yield condition. This is a generalization of a result due to R. Hill and M. P. L. Siebel [J. Mech. Phys. Solids, **1** (1953), 207–214]; this reference is not given in the paper, although the theory itself is quoted. The work of F. A. Gaydon and H. Nuttall [ibid. **6** (1957), 17–26] on rectangular sections might also have been mentioned for completeness.

This lower bound surface is in turn approximated from within by a polyhedron with 32 faces. With this a flow rule is associated by the plastic potential hypothesis. A corresponding exact solution for this substitute material is then obtained to the problem of a right-angle bent of square section, fixed at the ends and loaded obliquely at the midpoint of one leg. The full results and details of solutions are contained in an unpublished report and only certain parts appear here. The labour involved is considerable, although the authors claim it to be less than with ordinary upper and lower bound methods and that it gives an adequate approximation. *R. Hill* (Nottingham)

6822:

Jung, Hans. Über die Spannungen in dickwandigen Rohren bei elastisch-plastischer Beanspruchung. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **2** (1958), 251–256. (Russian and English summaries)

The author discusses stress-strain relations for work-hardening solids, obeying the yield criteria of von Mises and Tresca. Tresca's yield criterion is applied to a perfectly plastic thick-walled tube under internal pressure and to a tube with a linear strain-hardening law. The paper contains several misprints and undefined symbols.

W. T. Koiter (Delft)

6823:

Magiros, D. G. Lateral free vibrations of a simple prismatic bar in plasticity. I, II. Prakt. Akad. Athēnōn **31** (1956), 16–21, 168–171. (Greek. English summary)

The author outlines a method of solution of the differential equation of lateral free vibrations of prismatical bars when the moment-curvature relation is a nonlinear one. The solution is expressed with the help of an infinite

system of nonlinear integral equations. The uniform convergence of the solution as well as its stability are discussed. The method of successive approximations is used for solving a particular case.

A. Phillips (New Haven, Conn.)

6824:

Horvay, G.; Giaever, I.; and Mirabal, J. A. Thermal stresses in a heat-generating cylinder: the variational solution of a boundary layer problem in three-dimensional elasticity. Ing.-Arch. **27** (1959), 179–194.

The paper is concerned with various approximate forms of axi-symmetric stress functions for a semi-infinite circular cylinder, as determined by the direct methods of the calculus of variations and on the basis of an approximate separation of the variables r and z. Use is made of the functions obtained in this manner for the determination of thermal stresses in a semi-infinite cylinder with stress-free surface and a temperature distribution $T=k(1-r^2)$. *E. Reissner* (Cambridge, Mass.)

6825:

Eason, G.; and Sneddon, I. N. The dynamic stresses produced in elastic bodies by uneven heating. Proc. Roy. Soc. Edinburgh. Sect. A **65**, 143–176 (1959).

Following a brief exposition of the coupled field equations of classical thermoelasticity theory, the authors employ integral transform techniques to obtain formal solutions to two problems within this theory. The two problems concern the determination of the thermal deformations and stresses (A) in a solid occupying the entire space, and (B) in a semi-infinite solid bounded by a plane. In both problems the solid is under the influence of a prescribed (space and time-dependent) distribution of internal heat sources; problem (B), in addition, involves the specification of the surface temperature and of vanishing surface tractions.

The general solutions obtained to problems (A) and (B) are particularized to steady-state conditions and the analogous two-dimensional problems are considered as well. The solution to problem (A) is applied to study the effect of various time-dependent line sources of heat. The solution to problem (B) is illustrated with the example of a periodic line temperature on the surface of the half-space.

E. Sternberg (Providence, R.I.)

6826:

Gatewood, B. E. Thermal stresses in moderately thick elastic plates. J. Appl. Mech. **26** (1959), 432–436.

The three-dimensional stresses in the plate are investigated without using the plane-stress or plane-strain assumptions, the thickness of the plate being limited so that the normal stress in the thickness direction can be taken as a polynomial in the thickness variable. The temperature is taken as a polynomial in the thickness variable but with relatively large, though restricted, gradients with respect to the co-ordinates of the plane of the plate. (From author's summary)

E. H. Mansfield (Farnborough)

6827:

Ignaczak, Józef. Dynamic thermoelastic problem of a spherical cavity. Arch. Mech. Stos. **11** (1959), 399–408. (Polish and Russian summaries)

An infinite elastic space contains a spherical cavity. A heat source varying periodically with time is located at a generic point of the space. The author first determines the temperature field by superposing two solutions: one corresponding to a system of waves propagating from the point source, and another corresponding to a system reflected at the cavity. Together they produce zero temperature on the surface of the cavity. Then the system of thermal stresses is constructed as a superposition of particular solutions of the equations of motion of the space such that the surface of the cavity remains stress-free. Formal solution appears in the form of infinite series of Bessel and Legendre functions. Integration leads to solution for source varying arbitrarily with time.

The related problem with point source replaced by a time-dependent nucleus of thermoelastic strain is treated in an analogous manner.

No numerical values are given and no attempt is made to establish formal solutions. *H. Parkus* (Vienna)

6828:

Parkus, H. Spannungen beim Abkühlen einer Kugel. Ing.-Arch. **28** (1959), 251–254.

A sudden cooling of the sphere from the initial temperature T_0 to the surface temperature $T=0$ results for a sufficiently large temperature increment T_0 in the plastic flow of the material of the sphere. Three zones may be distinguished: (I) elastic nucleus, (II) plastic zone and (III) initially plastic outer zone which returned to the elastic state as the result of unloading. Further cooling leads to vanishing of the plastic zone (II).

The author succeeded in obtaining a rigorous solution in a closed form for the problem of elastic-plastic stresses, assuming incompressibility of the material.

Witold Nowacki (Warsaw)

6829:

Abir, David; and Nardo, S. V. Thermal buckling of circular cylindrical shells under circumferential temperature gradients. J. Aero/Space Sci. **26** (1959), 803–808.

The paper treats the buckling of a thin-walled circular cylindrical shell due to axial stresses which are constant axially but very arbitrary circumferentially. The analysis is exact within the framework of Donnell's equations.

E. H. Mansfield (Farnborough)

6830:

Nowacki, Witold. Steady-state thermal stresses in plates. Rozprawy Inż. **7** (1959), 3–24. (Polish. Russian and English summaries)

Temperature stresses are determined for an elastic plate of moderate thickness in which the temperature distribution in the direction of the thickness can be assumed as linear, while the distribution in the plane is arbitrary, continuous or discontinuous. Using Green's function of the differential equation of the problem, solutions are found for simple boundary conditions, such as infinite plate, plate-strip and a few cases of plates resting on an elastic foundation. *A. M. Freudenthal* (New York, N.Y.)

6831:

Frischbier, R. Spannungen und Verschiebungen in einem Kreiszylinder infolge unstetiger, aber ebener Temperaturverteilung. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **8** (1958/59), 387–392.

STRUCTURE OF MATTER

See also 6802, 6945, 7000, 7001, 7018.

6832:

Gross, Eugene P. Analytical methods in the theory of electron lattice interactions. I. Ann. Physics 8 (1959), 78–99.

In this paper, which is to be first in a series on this subject, the author attempts to lay down general methods for dealing with the interaction of an electron and a lattice by the methods of quantized field theory. The method of canonical transformations is used to treat the interaction for arbitrary values of the coupling constant. Special consideration is given to the development of general perturbation methods, and to the study of the ground state. *E. L. Hill* (Minneapolis, Minn.)

6833:

Kohn, Walter. Theory of Bloch electrons in a magnetic field: the effective Hamiltonian. Phys. Rev. (2) **115** (1959), 1460–1478.

The problem under consideration is a Bloch electron in a static magnetic field. It is then shown that for a non-degenerate band one can define an equivalent Hamiltonian from which the actual eigenstates can be calculated with an error that can be made arbitrarily small for small magnetic fields. This equivalent Hamiltonian has the following properties that make it particularly suitable for practical calculations: (1) It is a one-band-Hamiltonian, the interband matrix elements having been transformed away; (2) it depends only on the gauge-covariant operators; (3) it is periodic with the period of an arbitrary reciprocal lattice vector; (4) it can be written as a series in a set of functions which are completely symmetrized in the gauge-covariant, non-commuting operators. Detailed application of this Hamiltonian is promised in a forthcoming paper. *M. J. Moravcsik* (Livermore, Calif.)

6834:

Remnev, Yu. I. On the effect of irradiation on stresses and small deformations in solids. Soviet Physics. Dokl. **124** (4) (1959), 83–85 (540–541 Dokl. Akad. Nauk SSSR).

In this paper the connection between the stresses and small deformations in a crystalline solid which is expanded because of radioactive irradiation is studied. An approach to the determination of this expansion is given.

Werner Nowacki (Bern)

6835:

Gillis, Joseph; and Weiss, George. Nature of the singularities in the spectrum of a one-dimensional ionic lattice. Phys. Rev. (2) **115** (1959), 1520–1521.

It is known, by a theorem of van Hove, that regular lattices will have singularities in their frequency spectra, the type of singularity depending on the dimension. Although several studies of the one-dimensional lattice with Coulomb interactions have appeared, no one yet has succeeded in identifying the type of singularity that appears in addition to the inverse square root singularity, in the frequency spectrum. In this note we establish that for a one-dimensional crystal the singularity is of the form $g(\omega) \sim A\{(\omega-\omega_1)\times \ln[1/(\omega-\omega_1)]\}^{-1/2}$ where A is a constant and ω_1 is the position of the singularity.

Authors' summary

FLUID MECHANICS, ACOUSTICS

See also 6312, 6420, 6758, 6959, 7028, 7083, 7089.

6836a:

★Oswatitsch, K. Physikalische Grundlagen der Strömungslehre. Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I, (Mitherausgeber C. Truesdell), pp. 1–124. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vi+471 pp. DM 132.00.

6836b:

★Serrin, James. Mathematical principles of classical fluid mechanics. Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), pp. 125–263. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vi+471 pp. DM 132.00.

6836c:

★Howarth, L. Laminar boundary layers. Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), pp. 264–350. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vi+471 pp. DM 132.00.

6836d:

★Schlichting, H. Entstehung der Turbulenz. Handder Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), pp. 351–450. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vi+471 pp. DM 132.00.

The last edition of the fluid-dynamical section of the Handbuch der Physik which appeared about thirty years ago was a monumental affair which set a high standard in presentation and content, and which is still of great value. The new edition has accordingly been eagerly awaited in the expectation that it will carry on the tradition established. This hope has so far been amply realised. The four articles which constitute this part are of the same high standard as before, as indeed would have been expected from the names of the co-editor and authors. The articles have a breadth, depth and insight into the various branches of fluid dynamics covered, which fills the reader with admiration. Moreover, the argument is clear, elegant and brings out the vital features as well as could be desired. It should be an essential feature of the student, teacher or research worker's stock in trade {although the expense may deter personal possession}. There are however a few criticisms set out in the detailed reviews below.

The first article, by Dr. Oswatitsch, deals with the physical foundations of fluid dynamics, which may be interpreted as expressing the fundamental properties and mathematical ideas related to fluid flows in physical terms.

Dr. Oswatitsch begins by obtaining the fundamental equations in integral form and goes on to consider shock

waves and sound waves. Then he obtains the equations again, this time in differential form using the notion of spatial variables. In section IV he discusses some kinematical relations and examines their significance using cartesian coordinates only. This is followed by a consideration of the inviscid equations, by a section on vorticity, and by a section on internal friction. A number of different kinds of flows are considered in section VIII, beginning with a dimensional analysis of the governing equations and some general remarks on the significance of the controlling dimensional numbers—Reynolds, Prandtl, etc. Following this the significance of potential and stream functions is examined, which leads on to a discussion of the forces exerted by inviscid fluids. Potential flows form the subject of section XI both for incompressible fluids and the article concludes with a discussion on instability and some very brief remarks on turbulence.

The next article, by Professor Serrin, is concerned with basic mathematical ideas and theorems in continuous fluid mechanics. Many of these were of course developed many years ago and are to be found, for example, in Lamb's *Hydrodynamics*. However, much work has been done during the last thirty years in formalising the theory, in reappraising early contributions, extending results and proving new theorems. The article gives an account of these developments and also, by clothing many well-established proofs in the modern dress of vectors and tensors, brings out their essential elegance and power.

Professor Serrin begins with a discussion of the equation of motion in terms of spatial and of material variables, paving the way for the extensive use of both points of view to develop the basic ideas which is a characteristic and attractive feature of the article. Next he considers perfect fluids, establishing a number of well-known results such as Kelvin's circulation theorem and the property of vortex lines by particularly elegant methods. Section D is devoted to thermodynamics using Gibbs formulation, and the ideas developed are used in section E to discuss perfect gases on similar lines to the section on perfect fluids. Of particular interest here is the discussion of the Crocco-Vazsonyi equation and its application to homentropic and other flows {curiously for an otherwise highly logical work the author prefers isentropic to homentropic to describe a flow in which the entropy is constant everywhere}. There is also some account of special methods, or rather of the ideas underlying them in the study of perfect gas flows, including the hodograph method, variational principles, characteristics, sound waves and transonic flow. The basic properties of shock waves are discussed in section F, while section G is largely concerned with the general theory of viscous fluids including Stokesian fluids. It is a pity that room could not have been found here for Batchelor's theory of vorticity in regions with closed streamlines and for some remarks on limiting processes. For example, the final paragraph, on asymptotic behaviour, would have been improved by a detailed discussion of Stokes and Oseen flows.

Such criticisms as have been made in this review do not disturb the reviewer's overall view that Professor Serrin has produced an excellent article which should be made compulsory reading for all serious fluid dynamicists, particularly for those who teach and above all for those who intend to write a book themselves on the subject.

However, although these two articles considered separately are of a high quality it must be pointed out that collectively they are a little disappointing. In the first place there is a lot of duplication; old and well-established results such as Kelvin's circulation theorem are proved twice using virtually the same method. Second, the notations are not always the same, thus $\bar{\mu}$ and λ are both used for second coefficient of viscosity, although it is clearly desirable to have a uniform system of notation in this important series. Third, there are a number of important topics in the basic theory of fluid dynamics which have not really been covered. Of these the most important is turbulence which has hardly been given a hearing, but in addition hypersonic flow, unsteady flows and rotatory motions surely deserve comment. Even if it is felt that these topics are better discussed elsewhere, the saving of space in a unified article would have left room for a fuller discussion of the other topics in the book.

The next two articles are of quite a different kind from the others being mainly concerned with a particular type of flow namely that near a fixed solid body when the Reynolds number of the motion is large. The first of the articles is by Professor Howarth on laminary boundary layers. Restricted as this type of flow appears to be, it is of course of the utmost importance to the understanding of fluid flows in general. Indeed, so extensive are the researches into this aspect of fluid flow that the author has felt it necessary through limitation of space to omit such important topics as hypersonic boundary layers and to give only the briefest mention to others such as shock-waves boundary layer interaction.

The article begins with a statement of the fundamental ideas in incompressible boundary layers and a detailed discussion of the two-dimensional form: in this are included such aspects as exact solutions, series solution, separation, approximate methods, unsteady flows, shear layers, jets and flows up a step. Next, three-dimensional boundary layers are examined. Having derived the equations of motion the most serious difference between two and three dimensional problems is the notion of separation, and Professor Howarth simply and elegantly discusses its properties in the more complicated case. In section III the limitations of the boundary layer approximation are considered, including the notion of optimal coordinates. The author points out that the key difficulty in improving the boundary layer approximation to the flow past a flat plate lies in the neglect of the one of the highest order derivatives in the equation and rightly adds cautionary comments on the attempts he reports that had been made in this direction. For, since the article was written Imai has found the correct (and different) second approximation to the boundary layer solution. Section IV deals with the compressible boundary layer assuming that the Stokes relation between the two coefficients of viscosity holds: it is noted in passing that in fact this assumption is not strictly necessary since the second coefficient of viscosity never appears in the boundary layer equations. Finally, there are brief sections on heat transfer in incompressible flow and on boundary layer control.

This is a first rate article and the author is to be congratulated. Not only does he give the state of knowledge in boundary layer theory at the time of writing, but also brings it to life by soberly assessing the merits of the various contributions. The reader sees what makes the subject interesting and is stimulated to work in the field.

The final article, by Professor Schlichting, deals with the origin of turbulence. There are two major questions

involved in this problem: first, how is the laminar flow unstable and second, how does this instability lead to the irregular eddying motions characteristic of turbulent flow. So far as infinitesimal disturbances are concerned, the first question has now been answered in so many cases by theoretical arguments that that part of the problem may be regarded to a large extent as understood. For finite disturbances and for the answer to the second question we must still rely almost wholly on experimental methods and empirical arguments, but here too there have been great advances since the last edition of this handbook which are expertly presented in the article.

It begins with an account of the experimental evidence in the large of the onset of turbulence, including pipe flows and flow past bodies. Next, the theory of infinitesimal disturbances in a laminar flow is considered and applied to the flow past a semi-infinite plate and to plane Poiseuille flow {Poiseuille flow in a circular pipe is stable to small disturbances}. In section IV the theory is compared with experiments for neutral disturbances and the experimental evidence on the growth of turbulent spots discussed in detail. The effect of pressure gradients on transition follows in section V, using mainly 'similar' and Pohlhausen profiles.

Afterwards the effect of suction, of external influence such as rotation and wall curvature, of heat transfer and of compressibility in delaying or advancing transition are discussed mainly from the point of view of small disturbance theory. Since the existence of neutral small disturbances is neither necessary nor sufficient for transition to turbulence, and at best the amplified small disturbances do not lead immediately to the turbulent spots which are the precursors of transition, your reviewer would have welcomed more discussion on experimentally observed Reynolds numbers for transition. In particular, some remarks on transition properties at high Mach number would have been useful. In section IX three dimensional disturbances such as Taylor-Görtler vortices are considered and finally some experimental results on wall roughness are presented.

The only disappointment which I feel about this otherwise excellent article is that Professor Schlichting is loath to be critical, presenting each piece of work factually without sufficient discussion of assumptions, limitations, etc. Though the reader must look elsewhere for full critical assessment of the ideas there is no doubt that this article is a necessary starting point for a research worker in transition.

K. Stewartson (Durham)

6837:

★Rutherford, D. E. Fluid dynamics. University Mathematical Texts. Oliver and Boyd, Edinburgh-London; Interscience Publishers, Inc., New York; 1959. ix+226 pp. (4 plates) $1.95.

This is an elementary introduction to fluid dynamics intended for honours students. It gives an introductory account of the classical theory of incompressible flow in two dimensions and for flow with axial symmetry. Two further chapters give introductory accounts of the theory of compressible flow and of the effects of viscosity.

G. Temple (Oxford)

6838:

★Vallentine, H. R. Applied hydrodynamics. Butterworth & Co., Ltd., Toronto-London, 1959. viii+272 pp. $10.00.

The subject of this text-book is the application of the classical theory of inviscid, irrotational flow to real flows, and it develops the necessary concepts and mathematical methods without assuming more knowledge than may be possessed by an undergraduate reading mathematics or engineering. The first two chapters develop the basic theory of irrotational flow and examine the conditions in which the predictions of this theory will be applicable to the flow of fluids with viscosity. This involves a brief but, for the purpose, an adequate account of the general behaviour of viscous and turbulent flows. The remainder of the book describes methods for the computation of flow patterns, by numerical and graphical methods, by use of physical systems also obeying the Laplace equation, by the combination of standard flow patterns and by conformal transformations. The reviewer thinks that it fulfils its purpose as an introductory text-book for readers with interests in practical problems.

A. A. Townsend (Cambridge, England)

6839:

★Landau, L. D.; and Lifshitz, E. M. Fluid mechanics. Translated from the Russian by J. B. Sykes and W. H. Reid. Course of Theoretical Physics, Vol. 6. Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.; 1959. xii+536 pp. $14.50.

This is a translation of the first part of *Mehanika splošnyh sred*. The second edition of the whole work [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953] was reviewed in MR **16**, 412. The second part was translated under the title 'Theory of elasticity' and published separately [Course of Theoretical Physics, Vol. 7, 1959]; the translation is noted in MR **21** #5315.

The authors have contributed a special preface to this English edition and added a new chapter, on the general theory of hydrodynamic fluctuations. Topics not generally found in textbooks on the subject are treated in this work, such as heat transfer and diffusion, acoustics, combustion, superfluids and relativistic fluid dynamics.

6840:

★Sedov, L. I. Similarity and dimensional methods in mechanics. Translation by Morris Friedman (translation edited by Maurice Holt). Academic Press, New York-London, 1959. xvi+363 pp. (4 plates) $14.00.

This is a translation of the 4th edition of *Metody podobiya i razmernosti v mehanike* [Gosudarstv. Izdat. Tehn.-Teor. Lit, Moscow, 1957], reviewed in MR **19**, 898. The third (1954) edition was reviewed in MR **17**, 909 and the second (1951) in MR **14**, 809.

This translation had the benefit of the cooperation of the author, who also wrote a special foreword for it.

6841:

Krook, Max. Continuum equations in the dynamics of rarefied gases. J. Fluid Mech. **6** (1959), 523–541.

The fundamental ideas which must underlie any effort to improve the Navier-Stokes equations when the Knudsen number is finite are carefully considered. No specific flows are determined but the author proposes to

apply the ideas developed here to problems of shock-wave structure and Couette flow in later papers.

K. Stewartson (Durham)

6842:

Babič, Yu. A. Application of the Schwartz method to the solution of problems of subterranean hydraulics. Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Teh. Him. Nauk **1959**, no. 2, 61–66. (Russian. Azerbaijani summary)

6843:

Parsons, D. H. Contact transformations, of which the equation of axially symmetric liquid motion is a differential invariant. J. London Math. Soc. **34** (1959), 442–448.

It is established that Kelvin's transformation furnishes the only non-trivial contact transformation that leaves invariant the partial differential equations governing axially symmetric potential flow of an incompressible fluid.

P. R. Garabedian (New York, N.Y.)

6844:

Iacob, Caius. Sur la détermination du potentiel complexe de certains écoulements fluides à singularités données. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **9** (1957), 387–394. (Romanian. Russian and French summaries)

Es wird in einem in der komplexen Ebene gelegenen mehrfach zusammenhängenden Gebiet Ω, das von einer geschlossenen Kurve C_0 und p in ihrem Inneren gelegenen, einfach geschlossenen Kurven $C_1, \cdots, C_p$ begrenzt wird, eine Funktion $F_n(z_M, z_P)$ konstruiert, die bei festem $P \in \Omega$ folgende Bedingungen erfüllt: (1) F_n ist eine eindeutige analytische Funktion von z_M $(M \in \Omega)$; (2) $F_n(z_M, z_P) - (\alpha_n + i\beta_n)/(z_M - z_P)^n$ ist regulär in Ω (α_n, β_n reelle Konstanten); (3) der Imaginärteil von F_n ist gleich Null auf C_0 und gleich $f_j(P)$ (konstant) auf C_j $(j = 1, \cdots, p)$. $F_n(z_M, z_P)$ wird durch eine gewisse Hilfsfunktion ausgedrückt, die als Erweiterung der Greenschen Funktion des Gebietes Ω bezüglich $P \in \Omega$ aufgefaßt werden kann. Mit Hilfe der F_n kann in Ω das komplexe Potential einer inkompressiblen azyklischen Strömung dargestellt werden.

K. Maruhn (Giessen)

6845:

Viswanadha Sarma, L. V. K. Slow motion of a paraboloid of revolution in a rotating fluid. J. Fluid Mech. **3** (1958), 404–410.

"The slow uniform motion, after an impulsive start from relative rest, of a paraboloid of revolution along the axis of a rotating fluid is investigated by using a perturbation method. The principal purpose of the note is to illustrate the mechanism by which the fluid is not subjected to any substantial radial displacement, which is a direct consequence of the requirement that the circulation round material circuits should be constant when the perturbation velocities due to the motion of the paraboloid remain small. It appears that the mechanism is an oscillatory one in which the distance between any fluid particle and the axis of rotation oscillates sinusoidally in time with small amplitude. As time progresses, the amplitude of the oscillation decays to zero everywhere except on the paraboloid. The ultimate motion is then a rigid body rotation everywhere except on the paraboloid and the axis of rotation, where the perturbation velocities continue to oscillate indefinitely with small amplitude." (Author's summary)

K. Maruhn (Giessen)

6846:

Slepička, František. Berechnung der Potentialströmung für ein ebenes Spalt-Schaufelgitter. Apl. Mat. **4** (1959), 255–290. (Czech and Russian summaries)

Cascades of slotted airfoil sections can serve for obtaining very large turning angles in a flow without the risk of separation from the suction sides of the blades. Thus, the problem involved is mainly a boundary layer problem. The present investigation of the inviscid, two-dimensional and incompressible flow past cascades of infinitely thin slotted blades can be considered as a very first step towards the solution of a highly complex problem.

The singularity method—useful in many cascade problems—is convenient here too. The calculation of the velocities induced by the forward portions of the blades at locations on the aft portions and vice versa constitutes the new part compared with treatments already known.

The application of the method is illustrated by an example of a cascade similar to one for which experimental results are available. As only the optimum size of the slot (giving an 80% increase in lift with only a 20% increase in drag, as compared with an equivalent nonslotted section) was selected for the example, the results cannot be generalized, except for the conclusion that viscosity effects are the predominant ones.

M. Schilhansl (Cranston, R.I.)

6847:

Polášek, Jan. Eine Bemerkung zum Artikel von František Slepička: Berechnung der Potentialströmung für ein ebenes Spalt-Schaufelgitter. Apl. Mat. **4** (1959), 291–294. (Czech and Russian summaries)

It is shown in this short note that F. Slepička's [see above] method can be simplified by using functions of complex variables.

M. Schilhansl (Cranston, R.I.)

6848:

Vincenti, Walter G. Non-equilibrium flow over a wavy wall. J. Fluid Mech. **6** (1959), 481–496.

Extending work of Ackeret, the author discusses two-dimensional non-equilibrium flow over a wavy wall. The analysis is built around a partial differential equation of the third order which has the form

$$K \frac{\partial}{\partial x}[(1 - M_f^2)\phi_{xx} + \phi_{yy} + \phi_{zz}] + (1 - M_e^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0,$$

where ϕ is a perturbation velocity potential, K is a parameter involving the relaxation time, and M_f and M_e are, respectively, frozen and equilibrium Mach numbers. This equation is studied from the point of view of characteristics and singular perturbations, and a solution corresponding to the wavy wall problem is presented.

P. R. Garabedian (New York, N.Y.)

6849:

Gravalos, Fausto G. The dynamics of turbo-flow. Z. Angew. Math. Phys. **10** (1959), 347–380. (German summary)

The author formulates the problem of steady axially symmetric flow of an inviscid, incompressible fluid through

a turbo-machine. Force fields equivalent to a blade row are introduced and the connection between force field and vorticity examined for Lorenz and non-Lorenz flows. The dynamic characteristics of potential flow are analyzed and generalizations to von Mises flows and quasi-potential (the force field given as the product of the gradient of a scalar field and a second scalar field) flows are discussed. It is shown that for Lorenz flows the quasi-potential flows are also von Mises flows.

The quasi-potential flow problem is reduced to the solution of Stokes equation, $\psi_{rr}-r^{-1}\psi_r+\psi_{zz}=r\eta$, for the stream function ψ in terms of the vorticity along the tangential direction and a non-linear hyperbolic equation giving the blade shape. Some numerical examples are included. *R. C. MacCamy* (Pittsburgh, Pa.)

6850:

Carafoli, E.; et Săndulescu, Ş. Caractéristiques aérodynamiques des ailerons ayant un mouvement oscillatoire harmonique en régime supersonique. Rev. Méc. Appl. **4** (1959), no. 2, 201–227.

Neglecting terms of second order in frequency, the title problem is reduced to an equivalent problem in steady flow by well-known transformations. This steady flow problem is solved for wing-aileron configurations having polygonal boundaries by conical flow techniques.

J. W. Miles (Los Angeles, Calif.)

6851:

Bulygina, E. V. On the problem of a wing of a given volume with minimum wave drag. J. Appl. Math. Mech. **22** (1958), 1180–1184 (826–828 Prikl. Mat. Meh.).

The drag problem of a thin wing with thickness but no camber or incidence, in supersonic flow, is considered. Let $\alpha=\partial z/\partial x$, the slope of the surface in the stream direction. An integral formula for wave drag is given, attributed to M. N. Kogan, for the class of wings for which $\partial^2\alpha/\partial y^2$ is constant, y being the spanwise coordinate. The present note is an application of this formula to the isoperimetric problem stated in the title. The class of wings treated appears to be further restricted to those having similar profiles at all spanwise stations. The results appear in the form of a power-series (in x) expression for the profile shape. Application is made to a delta planform and it is concluded that the drag can be reduced 40% compared to a wedge profile. *W. R. Sears* (Ithaca, N.Y.)

6852:

Prosnak, W. J.; and Kucharczyk, P. The influence of the ground on the aerodynamic properties of an airfoil with jet flap. Arch. Mech. Stos. **11** (1959), 475–509. (Polish and Russian summaries)

The paper applies conventional treatment with the addition of a flow source suitably located on the airfoil contour. The complex potential for a circular profile is first determined. A transformation to a conventional zero thickness type airfoil is then applied so that a Kutta-Joukowsky condition may be used to establish the flow. The use of zero thickness permits a transformation independent of the ratio (dependent in other treatments) of chord length to ground distance. The computed circulation turns out to be a sum of two parts one of which is independent of the source. Specially selected integration contours and the so-called suction force are used to determine the aerodynamic coefficients. The drag coefficient turns out to be numerically equal to a jet coefficient $2Q/Uc$ in which $2Q$ is the flow quantity of the source and U, c are the free stream speed and chord length respectively. In order to reduce the results so that they are tractable to a general discussion of the influence of ground distance, the plausible simplifying assumption that the jet gap is small compared to the chord is made. It is shown in tables and plots that the lift fall off to a minimum as the airfoil approaches the ground and then increases to values substantially (over 50% available at the lower angles of attack) above the large ground distance values. The jet provides larger ground effect increases. *M. G. Scherberg* (Dayton, Ohio)

6853:

Hancock, G. J. Note on the extension of Evvard's method to wings with subsonic leading edges moving at supersonic speeds. Aero. Quart. **8** (1957), 87–102.

"Evvard's technique is applied to the problem of a thin finite wing moving at a supersonic speed when the leading edge is subsonic. It is developed in two methods: (i) in which the relationship between the pressure loading and the integrals of the downwash over the wing surface is extended as far as possible, and which has to be computed numerically; (ii) in which approximations are made for the upwash velocities in the neighbourhood of the leading edge, resulting in a series of standard integrals for the estimation of the pressure loading. Method (ii) is applied to the pressure loading on a flat plate triangular wing and cropped delta wing, and the application to more general shapes is discussed." (Author's summary) *K. Maruhn* (Giessen)

6854:

Ting, Lu. The pressure distribution on dihedral wings at supersonic speed. J. Fluid Mech. **6** (1959), 302–312.

Dans cet article, l'auteur donne une méthode approchée pour déterminer la répartition des pressions sur une aile à bords d'attaque supersoniques aplatie sur un dièdre, dans le cadre de la théorie linéaire. A cet effet il utilise essentiellement: (a) la continuité sur les lignes de Mach tracées sur l'aile à partir de l'apex avec les valeurs de la pression calculées sur les régions de l'aile où l'écoulement est bidimensionnel; (b) une formule exacte montrant que l'on peut relier la valeur du potentiel sur l'arête du dièdre avec celle obtenue pour une aile aplatie sur un plan; (c) une relation intégrale liant sur certaines lignes la distribution des pressions inconnues à celle connue des incidences.

Diverses applications sont exposées; dans le cas d'un écoulement conique, le résultat approché est comparé avec la situation exacte. *P. Germain* (Paris)

6855:

Keulegan, Garbis H. Energy dissipation in standing waves in rectangular basins. J. Fluid Mech. **6** (1959), 33–50.

6856:

Fontanet, Pierre. Sur la théorie de la génération de la houle par un batteur plan au deuxième ordre d'approximation (solution formelle). C. R. Acad. Sci. Paris **249** (1959), 1186–1188.

La théorie linéaire de la génération de la houle par un batteur plan est due à Havelock [Philos. Mag. (7) 8 (1929), 569–576]; l'auteur prolonge la théorie linéaire par le calcul de la seconde approximation.

La méthode des petits paramètres de Poincaré ramène le problème à la détermination de deux fonctions de trois variables. Ces fonctions qui vérifient un système d'équations aux dérivées partielles et des conditions aux limites sont explicitées au moyen de séries doubles et d'une fonction harmonique déterminée dans l'article suivant.

Cette étude apporte une contribution au problème de la génération des harmoniques. *H. Cabannes* (Marseille)

6857:

Fontanet, Pierre. Étude de la solution du second ordre du problème de la génération d'une houle cylindrique par un batteur plan. C. R. Acad. Sci. Paris **249** (1959), 1450–1452.

L'auteur détermine la fonction harmonique introduite dans la solution du problème précédent et en déduit, en particulier, que dans le canal à une certaine distance du batteur l'harmonique linéaire est engendrée par un effet du second ordre. Il établit ensuite la validité de la solution formelle construite dans la note précédente, en remarquant que les séries qui définissent cette solution se composent de termes qui contiennent un facteur exponential assurant la convergence absolue et uniforme dans un domaine convenable. *H. Cabannes* (Marseille)

6858:

Ponstein, J. Instability of rotating cylindrical jets. Appl. Sci. Res. A **8** (1959), 425–456.

The stability of solid and of hollow jets is discussed, including the effects of surface tension, and, in one case, viscosity. In certain cases non-rotationally symmetric perturbations are found to be more unstable than rotationally symmetric ones. Applications to the atomization of fuel are described, and for this purpose unstable perturbations are sought.

P. R. Garabedian (New York, N.Y.)

6859:

Gurevich, M. I. On the instability of certain jet flows with free surfaces. Soviet Physics. Dokl. **124** (4) (1959), 54–56 (998–1000 Dokl. Akad. Nauk SSSR).

The question of stability of free surface flows is analyzed by the usual perturbation technique. It is shown that the velocity potential can be neutrally stable in cases where the free surface itself is not quite stable. The example cited is that of uniform flow of incompressible, weightless fluid in a channel of finite depth. It is observed that inclusion of capillary forces stabilizes the flow.

P. R. Garabedian (New York, N.Y.)

6860:

Hachemeister, Charles A. The exact solution of Borda's mouthpiece in two dimensions. Quart. Appl. Math. **17** (1959), 299–304.

Evaluation in terms of elliptic integrals of the coefficient of contraction for a finite Borda mouthpiece.

P. R. Garabedian (New York, N.Y.)

6861:

Alekseevskii, V. P. On a certain problem in the theory of jets. J. Appl. Math. Mech. **22** (1958), 1192–1201 (833–838 Prikl. Mat. Meh.).

The problem considered is that of the impingement of two streams of an ideal incompressible fluid which move toward each other from infinity in the space bounded by solid walls. The motion is assumed to be steady, and symmetrical about the plane halfway between the fixed parallel walls. Using some methods of conformal mappings, a complete solution is given for this problem of plane flow, first for the case in which the bounded flow and the jet have the same density. Under certain assumptions this solution can be adapted to the case of different densities.

The analogous problem in the axially symmetric case is also considered. No complete solution is obtained, but certain relations are derived by use of the law of conservation of momentum which are useful in approximate solutions of the problem.

D. H. Hyers (Los Angeles, Calif.)

6862:

Kolberg, F. Untersuchung des Wellenwiderstandes von Schiffen auf flachem Wasser bei gleichförmig scherender Grundströmung. Z. Angew. Math. Mech. **39** (1959), 253–279. (English, French and Russian summaries)

The author discusses a ship moving with constant velocity in shallow water. The water is not at rest but is moving with uniform shear, that is, the velocity of the undisturbed fluid is in a direction parallel to the ship motion and has the form $az+b$, z measured upward from the free surface. The determination of the small perturbations in flow due to the ship is reduced to a potential theoretic problem which can be solved for certain ship approximations such as the Michell form. Formulas are obtained for the wave resistance and compared with those for motion through shallow water.

R. C. MacCamy (Pittsburgh, Pa.)

6863:

Ter-Krikorov, A. M. Exact solution of the problem of the motion of a vortex under the surface of a liquid. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 177–200. (Russian)

The paper deals with the steady two-dimensional motion, with velocity c, of a vortex, of strength Γ, below the surface of a heavy inviscid liquid, otherwise at rest, contained in a canal with a horizontal bottom at depth H below the undisturbed surface. By reducing the vortex to rest and mapping the region between the unknown free surface and the bottom on an infinite strip, the problem is made to depend upon the solution of a pair of non-linear integral equations, the essential parameters being c^2/gH and $\gamma=\Gamma/cH$. By going over to operators in Banach function spaces and by making certain estimates, the author proves that if $c^2>gH$ the square of the critical speed, and if γ is sufficiently small, the solution exists, is unique, and can be obtained by iterative methods.

L. M. Milne-Thomson (Madison, Wis.)

6864:

Gogolev, V. M. The approximate design of fluid current in a channel. Vestnik Leningrad. Univ. **12** (1957), no. 1, 197–200, 212–213. (Russian. English summary)

Der Verfasser untersucht die ebene Bewegung einer reibungslosen, wirbelfreien inkompressiblen Flüssigkeit in einem gekrümmten Kanal. Nach Einführung passender krummliniger Koordinaten wird die Strömung durch ein System von Integralgleichungen beschrieben, für deren Lösung ein Näherungsverfahren angegeben und an einem Beispiel durchgeführt wird. Es wird noch auf Erweiterungsmöglichkeiten, z.B. auf kompressible Unterschallströmung, hingewiesen. *K. Maruhn* (Giessen)

6865:

Gogolev, V. M. The calculation of liquid flows with axial symmetry in channels. Vestnik Leningrad. Univ. **13** (1958), no. 13, 147–156. (Russian. English summary)

In a previous paper [#6864 above] the author has considered a method of successive approximations for the calculation of plane flows of an ideal fluid in channels. In the present paper, by introducing suitable curvilinear coordinates, the author succeeds in extending this method to the case of axially symmetric flows in channels. Both compressible and incompressible flows are considered.

D. H. Hyers (Los Angeles, Calif.)

6866:

Babadžanyan, G. A.; and Nazaryan, A. G. A solution of a problem of plane laminar motion of a liquid in an open canal. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **12** (1959), no. 1, 61–73. (Russian. Armenian summary)

The mixed boundary-value problem for the Navier-Stokes equations indicated in the title is attacked by a perturbation scheme similar to Friedrichs' shallow-water theory. Expressions for velocity, pressure, and free-surface contour are given in zeroth and first order for an assigned ground profile and are specialized for a smooth bump. Questions of validity of the expansion scheme are not touched. *G. Kuerti* (Cleveland, Ohio)

6867:

Brenner, Howard. Pressure drop due to viscous flow through cylinders. J. Fluid Mech. **6** (1959), 542–546.

In the slow steady motion of viscous liquids through cylinders the fall in pressure between two planes is expressed as a sum of integrals; the planes must be such that the pressure across each is constant. In particular cases, such as the slow motion past a fixed sphere, the pressure fall can hence be found without detailed calculation of the flow. *W. R. Dean* (London)

6868:

Litvinkov, S. S. On a boundary value problem for the linearized equations of viscous flow. Dokl. Akad. Nauk SSSR **125** (1959), 998–1001 (Russian); translated as Soviet Physics. Dokl. **4**, 280–283.

Uniqueness and existence proofs are given in outline for the boundary value problem:

$$(1) \qquad \nabla^2\mathbf{u} + k^2\mathbf{u} = \frac{1}{\mu}\operatorname{grad} p, \quad \operatorname{div}\mathbf{u} = 0;$$

$\mathbf{u}$ prescribed on a closed boundary, and, for $r \to \infty$, $\mathbf{u} = O(r^{-1})$, $pu_r = o(r^{-1})$. (The equation is that of Stokes flow with exponential time dependence.) Uniqueness follows by reduction to the Helmholtz equation, after it has been shown that every $\mathbf{u}$ can be expanded in generalized spherical harmonics such that the elementary solutions satisfy the radiation condition [Vekua, same Dokl. **80** (1951), 341–343; MR **14**, 336]. The existence proof starts with the construction of a fundamental solution for (1), which is used to form, for both $\mathbf{u}$ and p, analogs of single and double layer potentials for vectors $\boldsymbol{\omega}$ and $\mathbf{g}$, respectively, prescribed on S. (These become Odquist's potentials if $k = 0$.) The jump conditions yield one integral equation for $\boldsymbol{\omega}$ and $\mathbf{g}$, which has a solution according to a theorem by S. G. Kreĭn [Uspehi Mat. Nauk **9** (1954), no. 3 (61), 149–153; MR **16**, 262].

G. Kuerti (Cleveland, Ohio)

6869:

Levey, H. C. Heat transfer in slip flow at low Reynolds number. J. Fluid Mech. **6** (1959), 385–391.

The convective heat transfer from a cylinder, such as a hot-wire, in two-dimensional gas flow at low Mach and Reynolds numbers is studied. The relative temperature differences are assumed small, and so is M^2/R. An Oseen approximation for the temperature field, independent of the velocity field, then becomes possible, and a reasonable temperature jump condition is shown to lead to a heat transfer result extending the known continuum and molecular results to arbitrary Knudsen numbers.

R. E. Meyer (Providence, R.I.)

6870:

Stănescu, Cristian. La méthode de Moskhelichvili dans les mouvements plans des fluides visqueux incompressibles. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **9** (1957), 395–414. (Romanian. Russian and French summaries)

Die Bestimmung der Stromfunktion ψ der "langsamen" ebenen stationären Bewegung einer inkompressiblen zähen Flüssigkeit führt bei Vernachlässigung der Terme höherer Ordnung bezüglich der Geschwindigkeit und ihrer Ableitungen bekanntlich auf die biharmonische Gleichung $\Delta\Delta\psi = 0$. Wie man weiß, lassen sich Geschwindigkeitskomponenten, Druck und Wirbelstärke mittels der Lösung von Goursat: $\psi = \operatorname{Re}[\bar{z}\varphi_1(z) + \chi(z)]$ (φ_1, $\chi = \int \varphi_2(z)dz$ analytisch), ausdrücken. Nach einer allgemeinen Analyse der Randbedingungen untersucht Verf. die Bewegungen in mehrfach zusammenhängenden Gebieten und in Außengebieten rotierender Bereiche (mit Beispielen).

K. Maruhn (Giessen)

6871:

Săvulescu, Şt. N. Betrachtungen über einige Lösungen des Falles der instationären inkompressiblen Grenzschicht. Rev. Méc. Appl. **4** (1959), no. 1, 7–18.

A translation of Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 867–879 [MR **20** #6257]. The word "inkompressiblen" in the title appears to be a misprint for "kompressiblen".

6872:

Yang, Kwang-Tzu. Unsteady laminar compressible boundary layers on an infinite plate with suction or injection. J. Aero/Space Sci. **26** (1959), 653–662.

Using standard assumptions and techniques the momentum and energy equations of the compressible boundary layer are reduced to their incompressible form.

Approximate methods using the momentum integral and two different forms of the velocity profile are developed to determine the skin friction at the plate. The results are compared with known exact solutions when the velocity of the plate and the suction velocity are effectively certain powers of the time.

The heat transfer is also examined in a similar way.

K. Stewartson (Durham)

6873:

Bethe, Hans A.; and Adams, Mac C. A theory for the ablation of glassy materials. J. Aero/Space Sci. **26** (1959), 321–328.

Glassy materials, having a viscosity which increases exponentially as the temperature decreases, form a flowing liquid layer when subjected to aerodynamic heat transfer and shear forces. The analysis of their ablation is restricted to the stagnation region of a blunt body. The solution is found for the liquid layer with the gas boundary-layer assumed known. Next the coupling between the liquid and gas boundary-layers is considered, first for the case of no evaporation and then for partial evaporation of the ablating liquid. Only when vaporisation takes place does the liquid layer induce any significant change in the heat transfer and shear forces in the gas. The analysis shows that the effective energy of ablation for glassy materials depends primarily upon flight speed and is insensitive to altitude. If the stagnation viscosity in the liquid layer is very large practically all the liquid evaporates. Some numerical results are given for Pyrex.

D. C. Pack (Glasgow)

6874:

Walz, Alfred. Compressible turbulent boundary layers with heat transfer and pressure gradient in flow direction. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 53–70.

The author expresses the partial differential equation for the thermal boundary layer in two dimensional flow as an infinite set of ordinary differential equations for partial integrals of the flow. An approximate empirical form for the turbulent velocity profile is constrained to satisfy the first momentum integral condition and the first energy integral condition of the infinite set. The author states that "Nevertheless results obtained are of sufficient accuracy for engineering purposes."

W. V. R. Malkus (Woods Hole, Mass.)

6875:

Ullmann, D. Untersuchungen zu einer neuen Theorie der Turbulenzentstehung. Ann. Physik (7) **3** (1959), 316–322.

The author investigates the stability of two-dimensional parallel flow in a channel with respect to a two-dimensional time-dependent disturbance described as an acoustic wave. The basic flow is the standard one for an incompressible fluid, with a parabolic velocity distribution and a linear pressure variation. The two velocity components and the pressure of the total flow (basic flow plus disturbed flow) are assumed to satisfy two dynamical equations of the usual form for incompressible flow (except for an unexplained modification of the viscous terms) and the continuity equation for compressible flow, with density derivatives replaced by pressure derivatives by use of the isentropic pressure-density relation. The energy equation is not considered, and no justification is given for the neglect of heat conduction and the assumption of constant entropy. With the specification of special analytical forms for the velocity and pressure disturbances, some analytical manipulations and further assumptions lead to a minimum critical Reynolds number for unstable disturbances. {Reviewer's note: Contrary to a statement of the author, previous investigations based entirely on the equations for incompressible flow have predicted instability for the two-dimensional flow in a channel (plane Poiseuille flow). It is the corresponding axisymmetric flow for which instability has not been demonstrated [for example, see the book by C. C. Lin, *The Theory of Hydrodynamic Stability*, University Press, Cambridge, 1955; MR **17**, 1022].}

D. W. Dunn (Ottawa, Ont.)

6876:

Schaefer, John W.; and Eskinazi, Salamon. An analysis of the vortex street generated in a viscous fluid. J. Fluid Mech. **6** (1959), 241–260.

Hot-wire measurements are reported of velocity fluctuations in the vortex street behind a piano wire in a low-turbulence wind-tunnel. The data are fitted to a kinematical model, resembling that of S. G. Hooker [Proc. Roy. Soc. London Ser. A **154** (1936), 67–89]. In the new model, the transverse distance h_n of the center of the nth vortex from the centerline is fitted to the data; the spread of vorticity about this center is deduced from the diffusion equation for vorticity. Since h_1, h_2, h_3, $\cdots$ as well as the vortex frequency and strength are taken as disposable parameters, it is not too surprising that the model fits the data to $\pm 20\%$. Ratios h/a between 0.07 and 0.3 are reported; in Hooker's data, the reported ratios were 0.28–0.5.

G. Birkhoff (Cambridge, Mass.)

6877:

Oser, Hansjörg. Erzwungene Schwingungen in rotierenden Flüssigkeiten. Arch. Rational Mech. Anal. **1** (1957), 81–96.

In einer gleichförmig rotierenden, reibungsfreien, inkompressiblen Flüssigkeit führe eine Kreisplatte, die senkrecht zur Rotationsachse steht, in dieser harmonische Schwingungen aus. Die durch einen passenden Lösungsansatz aus den linearisierten Eulerschen Bewegungsgleichungen sich ergebenden linearen partiellen Differentialgleichungen 2. Ordnung für die Amplituden der Geschwindigkeitskomponenten und des Druckes ändern bei Durchschreiten einer gewissen kritischen Frequenz ihren Typus. Verfasser betrachtet (im Meridianschnitt) die im hyperbolischen Fall auftretenden Charakteristiken, die das Strömungsbild in Bereiche verschiedenartiger Lösungen aufteilen, und erwähnt, daß ihm auch der experimentelle Nachweis dieser Charakteristiken gelang. Geschwindigkeit und Druck werden berechnet und insbesondere in der Umgebung der Charakteristiken untersucht.

K. Maruhn (Giessen)

6878:

★Chandrasekhar, S. Variational methods in hydrodynamics. Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 139–141. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

Der Verfasser erläutert an zwei Beispielen von Eigenwertproblemen, die sich bei hydrodynamischen Stabilitätsuntersuchungen ergeben, die Zurückführung auf die Lösung mit Hilfe von Variationsmethoden.

K. Maruhn (Giessen)

6879:

Görtler, Henry. Einige neuere Ergebnisse zur hydrodynamischen Stabilitätstheorie. Z. Flugwiss. **8** (1960), 1–8. (English and French summaries)

A brief presentation is given of the theoretical research work carried out at Freiburg/Breisgau during the recent years on the theory of hydrodynamic stability. The report addresses itself to the interested non-specialists and lays emphasis on the physical side, whilst touching the mathematical complexities only in passing. The investigations treated here are aimed at making contributions to a deepened and more general understanding of the mechanism of the transition from laminar to turbulent flows.

Author's summary

6880:

Feldman, Saul. On the instability theory of the melted surface of an ablating body when entering the atmosphere. J. Fluid Mech. **6** (1959), 131–155. (2 plates)

This paper investigates the instability of an ablating body in a vacuum with two dimensional disturbances in the direction perpendicular to the undisturbed flow. Two cases are considered: (1) the liquid layer is semi-infinite in extent and the viscosity is an exponential function of the distance away from the interface, and (2) the layer has constant viscosity but finite thickness. In the first case the disturbance equation is solved for some limiting problems of large and small nR or α, where R is the Reynolds number, n is the amplification exponent and α is the wave number. The solution of the second case is given in graphic form. Some limiting problems are compared to those of the first case. Finally, a numerical example and some remarks on the cause of disturbances during a decelerating flow are given.

L. N. Tao (Chicago, Ill.)

6881:

Lyon, Richard H. On the diffusion of sound waves in a turbulent atmosphere. J. Acoust. Soc. Amer. **31** (1959), 1176–1182.

A theoretical analysis is made of the directional and frequency diffusion of a plane monochromatic sound wave in turbulence. The turbulence is assumed to be statistically homogeneous, isotropic and stationary, and the treatment is based on the diffusion equation for the energy density of sound waves and employs the scattering cross section derived by Kraichnan.

The various simplifying assumptions that were made are discussed in detail, and it is suggested that the results probably apply to the propagation of sound at aircraft altitudes.

The rate of spread of the incident wave in frequency and direction is calculated.

The power spectrum and autocorrelation function are given. The auto-correlation function is a cosine curve at the sound frequency damped by a factor of the form $\exp(-b^2(\omega\tau)^2)$, where b^2 is a function of the parameters of the turbulence and depends linearly on the distance from the source.

W. J. Pierson, Jr. (New York, N.Y.)

6882:

Loitsianskii, L. G. The hypothesis of localness in the turbulent motion of a viscous fluid. J. Appl. Math. Mech. **22** (1958), 844–859 (600–611 Prikl. Mat. Meh.).

To support the differential or "local" approach to the semi-empirical study of turbulent momentum transport, the author shows that inclusion of the molecular transfer term in the Prandtl mixing length formula permits the determination of a continuous velocity profile from the boundary to the logarithmic range. Two empirical parameters are required as in the usual formulation. The author does not complete his argument that a similar continuous velocity profile can be found from the von Kármán hypothesis, where local transport is related to the first two local derivatives of the velocity. In the opinion of the reviewer the mixing length approach is clearly not "local" when this length is taken as proportional to the distance from the boundary. The von Kármán approach is "local". If both give similar velocity profiles, what has one learned about the turbulent process?

W. V. R. Malkus (Woods Hole, Mass.)

6883:

Dorfman, A. Sh. Computation of a Laval nozzle. J. Appl. Math. Mech. **22** (1958), 554–561 (399–404 Prikl. Mat. Meh.).

An analytical solution is found for the flow in the initial supersonic part (up to Mach number 1.5) of a shock-free plane-parallel Laval nozzle, by use of a suitable approximation to the adiabatic relation in Chaplygin's hodograph equations. The stream function and its derivatives are shown to have series expansions on the sonic line similar in form to those obtained from Frankl's [Izv. Akad. Nauk. SSSR Ser. Mat. **9** (1945), 387–422; MR 8, 416] expressions for the solution in the subsonic region.

A first approximation to the solution recovers Fal'kovich's solution [Akad. Nauk SSSR. Prikl. Mat. Meh. **10** (1946), 503–512; MR 8, 416].

D. C. Pack (Glasgow)

6884:

Dumitrescu, Lucian. Minimum length of a shock-tube, for given working times. Rev. Méc. Appl. **2** (1957), no. 1, 61–77.

A calculation is made of the minimum length of shock-tube which will permit a given test time at a prescribed Mach number, for a test made either ahead of or behind the contact discontinuity.

D. C. Pack (Glasgow)

6885:

Filimon, Ioan. Sur les corrections de compressibilité aux mouvements continus subsoniques. Cas de l'obstacle elliptique. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 467–504. (Romanian. Russian and French summaries)

Der Verfasser entwickelt ein Verfahren, um in einer Z-Ebene die stationäre rotationsfreie Unterschallströmung um eine gegebene konvexe Kontur Γ unter Zugrundelegung der Tschapliginschen Dichte-Geschwindigkeitsrelation $\rho=\rho_0(1+AV^2)^{-1/2}$ (A konstant) zu bestimmen. Hierzu benötigt man daneben noch in einer z-Ebene eine entsprechende Strömung (Potential $f(z)$) um eine vorerst unbekannte Kontur Γ_1. Der Zusammenhang mit der

Z-Ebene lautet $dZ=C_1dz+C_2\overline{(df/dz)}^2dz$ (C_1, C_2 aerodynamische Konstanten). Zur Herstellung der konformen Abbildung des Äußeren von Γ_1 auf das Äußere des Einheitskreises, die man für die gesuchte Geschwindigkeitsverteilung braucht, ergibt sich ein nichtlineares Randwertproblem, dessen Lösbarkeit auf Grund allgemeiner Sätze feststeht. Die Lösung wird in Gestalt einer Potenzreihe nach einem Parameter ν von der Größenordnung M_∞^2 (M_∞ Machzahl im Unendlichen) angesetzt, für deren Koeffizienten sich Funktionalgleichungen ergeben. Diejenige für den von ν freien Term läßt sich lösen, wenn die konforme Abbildung des Äußeren der bekannten Kontur Γ auf das Äußere des Einheitskreises bekannt ist; die Bestimmung der übrigen Koeffizienten führt auf lineare Probleme, deren Lösung sich in gewissen Fällen auf Quadraturen zurückführen läßt.—Diese Betrachtungen werden dann auf eine elliptische Kontur Γ unter Beschränkung auf einen in ν linearen Ansatz angewendet. Für einige Achsenverhältnisse werden bei verschiedenen Werten von M_∞ die Maximalgeschwindigkeiten auf der Ellipse sowie die entsprechenden kritischen Machzahlen berechnet.—Schließlich wird noch die Kontur Γ_1 der z-Ebene als Ellipse vorgegeben, was zu einer entsprechenden kompressiblen Strömung um eine von einer Ellipse abweichenden Kontur Γ führt. *K. Maruhn* (Giessen)

6886:

Filimon, Ioan. Sur les corrections de compressibilité aux mouvements continus subsoniques autour d'un obstacle donné. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **9** (1957), 433–449. (Romanian. Russian and French summaries)

Enthält im wesentlichen den theoretischen Teil der im vorhergeheden Referat besprochenen Arbeit.

K. Maruhn (Giessen)

6887:

Tricomi, Francesco G. Sull'equazione di Tomotika e Tamada. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 135–139.

This note shows how to derive a large class of special solutions to the mixed type equation

$$(1-y^2)\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 x}{\partial y^2}+y\frac{\partial x}{\partial y}=0.$$

Such solutions have been used by Tomatika and Tamada and by Nocilla to solve transonic flow problems. By introducing the variables $x=\text{arc tg}\,(\eta/\xi)-\eta+\text{const}$, $y=(\xi^2+\eta^2)^{1/2}$ the given equation is shown to have solutions of the form $f(\xi)+y(\eta)$. Since the derivative with respect to x of any solution of the given equation is again a solution, a still larger class of solutions can be found by integrating and differentiating with respect to x.

C. S. Morawetz (New York, N.Y.)

6888:

Iur'ev, I. M. On the theory of plane gas flows. J. Appl. Math. Mech. **23** (1959), 287–299 (201–208 Prikl. Mat. Meh.).

Utilisant les variables canoniques de l'hodographe, l'auteur établit une correspondance entre les équations régissant les écoulements de fluides compressibles ayant des éjections différentes. La correspondance faisant intervenir deux constantes arbitraires, on peut ainsi espérer par itération approcher sensiblement la loi en fluide réel à partir d'un gaz fictif pour lequel on sait résoudre explicitement le problème considéré. Une application est faite à l'étude d'une tuyère en reprise transonique. Le Principe de l'application d'une telle transformation à la mécanique des fluides a été donnée par Peres [C. R. Acad. Sci. Paris **219** (1944), 501–504; MR **7**, 342]; des extensions analogues —"transoniques"—ont fait l'objet de nombreux travaux, en particulier C. Löwner, NACA Tech. Note **TN 2065** (1950) [MR **13**, 464] et P. Germain et M. Fenain, C. R. Acad. Sci. Paris **234** (1952), 592–594 [MR **13**, 793].

P. Germain (Paris)

6889:

Clarke, Joseph H. Reverse flow and supersonic interference. J. Fluid Mech. **6** (1959), 272–288.

6890:

Šmyglevskiĭ, Yu. D. Bodies of rotation having minimal resistance at supersonic velocities. Dokl. Akad. Nauk SSSR **126** (1959), 958–960 (Russian); translated as Soviet Physics. Dokl. **4**, 541–543.

6891:

Larisch, E. Interactions of detonation waves. J. Fluid Mech. **6** (1959), 392–400.

The properties of detonation waves at an oblique angle to a stream of combustible gas are investigated by the use of detonation polars which are closely analogous to the well known forms of shock polar. The geometric properties of the detonation polars that are established are used to consider qualitatively the various possible interactions between detonation waves, shock and expansion waves and shear discontinuities with due allowance for the fact that the strength of shock waves is limited by stability considerations. A brief indication is given as to how a quantitative analysis of such interactions may be performed.

J. J. Mahony (Sydney)

6892:

Egorov, I. T. Oblique shock against a fluid. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 8, 41–47. (Russian)

Das Eindringen einer ebenen Platte aus der Luft unter einem festen Winkel in eine reibungsfreie, inkompressible Flüssigkeit wird in Analogie zum Problem der instationären Bewegung einer unendlichen Flüssigkeit um ein dünnes Profil gesetzt (ohne Totwasser). Unter der Annahme, daß die Richtung des Eindringens genügend wenig von der Horizontalen abweicht, wird für die Lösung der singulären Volterra'schen Integralgleichung erster Art ein Näherungsverfahren angegeben und an einem Beispiel erläutert. *K. Maruhn* (Giessen)

6893:

Gubanov, A. I. Reflection and refraction of shock waves at the interface between two media. Ž. Tehn. Fiz. **29** (1959), 615–624 (Russian); translated as Soviet Physics. Tech. Phys. **4**, 549–557.

"We obtain the general equations from which one may find the pressure in the reflected shock wave when an

oblique plane wave is incident on a plane boundary separating two arbitrary media. When there is either a small or a large difference in the two media, this equation can be solved approximately for the case of a shock wave of small magnitude." *Author's summary*

6894:

Ryzhov, O. S.; and Khristianovich, S. A. On nonlinear reflection of weak shock waves. J. Appl. Math. Mech. **22** (1958), 826–843 (586–599 Prikl. Mat. Meh.).

An analysis is presented for a phenomenon which the authors describe as 'short waves'. Such a wave is a narrow region in the flow behind a weak shock wave in which the pressure gradients are so large that the propagation of the shock wave is not governed by the usual acoustic approximation but can be determined only with due allowance made for the interaction between the shock and short wave. The general non-linear equations which represent a valid approximation to the full equations are derived and particular solutions of these approximate equations are obtained by similarity methods.

The problem of reflexion of weak shock waves from a rigid plane surface is then considered for the case when the angle of incidence is near the critical value for the breakdown of regular reflexion. In this problem short waves exist close to the point of reflexion but the boundary conditions are not satisfied by any of the class of similarity solutions. The authors take as their solution the one which most nearly satisfies an equation of conservation of momentum. *J. J. Mahony* (Sydney)

6895:

Bianco, Edmond; Cabannes, Henri; et Kuntzmann, Jean. Sur les nombres de Crocco pour les écoulements de révolution. C. R. Acad. Sci. Paris **249** (1959), 1312–1314.

Le nombre de Crocco, c'est à dire le nombre de Mach amont pour lequel le rayon de courbure de l'onde de choc attachée à la pointe d'une ogive de révolution est nul, est calculé pour $\gamma = 1, 4$ en fonction de l'angle d'ouverture de cône. *J. P. Guiraud* (Meudon)

6896a:

Roever, W. L.; and Vining, T. F. Propagation of elastic wave motion from an impulsive source along a fluid/solid interface. I. Experimental pressure response. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 455–465, 509–523.

6896b:

Strick, E. Propagation of elastic wave motion from an impulsive source along a fluid/solid interface. II. Theoretical pressure response. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 455–456, 465–488, 509–523.

6896c:

Strick, E. Propagation of elastic wave motion from an impulsive source along a fluid/solid interface. III. The pseudo-Rayleigh wave. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 455–456, 488–523.

The three parts of this paper deal with various aspects of the interaction across a plane surface of separation, between acoustic waves propagated in a fluid and elastic (dilatation and shear) waves propagated in a solid. The aim is to examine the effects of setting the sound velocity in the fluid to be larger or smaller than the shear wave velocity in the solid. The first part is a description of the equipment and the experimental work required to obtain some data on this effect. The pressure response at a fluid/solid interface due to a spark source close to the solid surface is measured and described by Messrs. Roever and Vining.

In the second part, Strick carries out the theoretical work on the linearized propagation problem, in order to find the pressure and velocity response for an impulsive source, and he uses Cagniard's method to find the exact solution for the two-dimensional problem with an impulsive line source for the purpose of comparison with experiment. The method is essentially one for inverting a complicated Laplace transform.

In the third part Strick examines certain aspects of his solution in detail, particularly the nature of the surface wave to be expected. It will be remembered that, in the complete absence of fluid, there can be set up on the surface of an elastic solid, a non-dispersive Rayleigh wave with a characteristic velocity somewhat smaller than the shear wave velocity. In the presence of fluid, however tenuous, there is a corresponding non-dispersive wave which Strick calls, rather confusingly, a Stoneley wave, whose characteristic velocity must be less than the smaller of the sound and shear velocities. Thus if the fluid of small density has a sound velocity which is smaller than the Rayleigh velocity for the solid alone, the change, as far as the velocity of the surface wave is concerned, from fluid to no fluid is a non-uniform one.

However, in such a case, in the presence of fluid a dispersive wave which travels at about the true Rayleigh wave velocity may be found, and this is the main topic of part III. It is not a clear presentation because Strick likens this disturbance to a radiating wave, or to a critically refracted dilatation wave.

These descriptions may not seem unreasonable when the source is off the interface, but since the same effect is present when the source is on the interface it is then a simple matter to pick out the critically refracted waves and to show their irrelevance in the matter of surface waves. Likewise, in the strictly transient problem considered, there is no such thing as a radiated wave; this can only appear as a contribution to the field when a source of harmonic time dependence is examined.

In fact, for a line source on the interface there is a line of stagnation moving with the true Rayleigh velocity. The rate of change close to this stagnation line of either the pressure change or the velocity components in the fluid becomes very large for small values of fluid density, but the associated maxima and minima do not share any characteristic velocity.

The problem gives a good qualitative explanation for the existence of sound waves associated with distant explosions. *V. M. Papadopoulos* (Providence, R.I.)

6897:

Brekhovskikh, L. M. On the attenuation of Rayleigh waves during propagation along an uneven surface. Soviet Physics. Dokl. **124** (4) (1959), 150–153 (1018–1021 Dokl. Akad. Nauk SSSR).

Using a successive approximation method the author obtains an expression for the attenuation coefficient of Rayleigh waves propagating along an uneven surface, where it is assumed that the depths of the surface irregularities are small compared to the wavelength. The disturbance is represented by a double Fourier series and numerical results are displayed for a one-dimensional irregularity in three cases, viz. aluminium, the earth's crust and steel. *M. H. Rogers* (Bristol)

6898:

Kanevskiĭ, I. N. Analysis of the diffraction of a converging cylindrical wave by a cylinder. Akust. Ž. **5** (1959), 151–156 (Russian); translated as Soviet Physics. Acoust. **5**, 152–157.

A concave cylindrical wave of included angle $2\alpha_m$ converges on a coaxial rigid cylinder. The incident and scattered waves are represented, respectively, by series expansions in Bessel and Hankel functions of the first kind. An asymptotic expression is obtained for the ratio of scattered intensity at large distances from a slender cylinder to the intensity incident on the cylinder's surface. This is compared with a similar expression for plane-wave scattering. The effective scattering cross section and the effective periodic force per unit length of cylinder are derived. *W. W. Soroka* (Berkeley, Calif.)

6899:

Saharova, M. P. Asymptotic representation of the sound field of a point source in a wedge-shaped region. Akust. Ž. **5** (1959), 215–220 (Russian); translated as Soviet Physics. Acoust. **5**, 214–219.

A representation of the distant field in a narrow wedge-shaped region of a harmonic point source, also situated in the region, is obtained from Sommerfeld's integral solution by simple and obvious approximations. For a source on the edge, this field is studied near the edge in spherical coordinates, and it is concluded that the field attenuates exponentially, the exponent containing the reciprocal of the wedge angle.

R. N. Goss (San Diego, Calif.)

6900:

Paul, David I. Wave propagation in acoustics using the saddle point method. J. Math. Phys. **38** (1959/60), 1–15.

Using the saddle point method of Baños and Wesley [Scripps Inst. Oceanography, Univ. Calif., Ref. **53**-33, 1953; Ref. **54**-31, 1954] for the asymptotic evaluation of the fields radiated by a point source in the presence of two semi-infinite isotropic media, the author extends the range of validity of his previous results [J. Acoust. Soc. Amer. **29** (1957), 1102–1109; MR **19**, 800].

L. B. Felsen (Brooklyn, N.Y.)

6901:

Lyamshev, L. M. A question in connection with the principle of reciprocity in acoustics. Dokl. Akad. Nauk SSSR **125** (1959), 1231–1234 (Russian); translated as Soviet Physics. Dokl. **4**, 406–409.

The reciprocity principle is formulated in sufficiently general terms to be valid for an arbitrary set of harmonic sources distributed throughout a volume in which also is found any combination of elastic objects (e.g., plates, shells, rods, membranes), on which any system of external forces is acting. The source distribution may be random provided its space correlation function is available. It is shown that certain relations known to exist among the solutions of radiation and diffraction problems are immediate consequences of this principle.

R. N. Goss (San Diego, Calif.)

6902:

Fay, R. D. Spherical waves of finite amplitude. J. Acoust. Soc. Amer. **31** (1959), 1377–1379.

The theory developed by the author [same J. **28** (1956), 910–914; **29** (1957), 1200–1203; MR **19**, 1009] of transmission of plane waves of finite amplitude is extended to the analysis of finite spherical waves. The basic assumption is that the conical paths of transmission can be represented by a series of cylinders of infinitesimal length with infinitesimal increments in the area at the junctions.

R. N. Goss (San Diego, Calif.)

6903:

Yankov, V. V. Behavior of a conducting gaseous sphere in a quasi-stationary electromagnetic field. Soviet Physics. JETP **36** (**9**) (1959), 388–391 (560–564 Ž. Éksper. Teoret. Fiz.).

The author investigates the stability of a homogeneous plasma sphere of infinite conductivity against small deformations. The sphere is immersed in a uniform isotropic radiation field. The time average potential energy of the system in a given state of deformation consists of two parts: the average electromagnetic potential energy of the sphere in an external field; the work done against the radiation pressure while adiabatically changing the volume of the sphere (considered as an ideal gas) from its equilibrium volume to its actual volume. The author now supposes that even in a stationary state the stability is determined by the sign of the average energy difference between the equilibrium state and the state of small deformation.

The results are as follows. If the external isotropic field is purely electric there is no equilibrium; if it is purely magnetic the plasma sphere is in stable equilibrium against any small, arbitrary deformation. In the general case equilibrium can exist only if the mean square magnetic field is larger than twice the mean square electric field. The equilibrium is stable if the order of the spherical harmonics describing the deformation satisfies an inequality. This inequality roughly states that for stability the number of nodal curves having an axis of symmetry must be less than the ratio of the mean square magnetic field to the mean square electric field. The author also discusses the extension of his results for slightly inhomogeneous external fields. *N. L. Balazs* (Princeton, N.J.)

6904:

Žigulev, V. N. On the phenomenon of magnetic "detachment" of the flow of a conducting medium. Dokl. Akad. Nauk SSSR **126** (1959), 521–523 (Russian); translated as Soviet Physics. Dokl. **4**, 514–516.

6905:

Zhigulev, V. N. Theory of the magnetic boundary layer. Soviet Physics. Dokl. **124** (**4**) (1959), 57–60 (1001–1004 Dokl. Akad. Nauk SSSR).

This is a study of the penetration of magnetic field and electric current into a fluid of large electrical conductivity from an adjacent solid surface. A plane boundary is considered along which the fluid moves and in which an electric current flows, either (i) at right angles to or (ii) parallel to the direction of fluid flow. In both cases the result is a layer of thickness of order $LR^{-1/2}$, where L is the distance from the leading edge of the boundary and R is the magnetic Reynolds number based on L, in which the current density drops off rapidly from the boundary value to zero. These layers are called magnetic boundary layers. Across such layers the total pressure does not change appreciably; hence the real pressure changes markedly.

Self-similar solutions appear in cases analogous to the well-known Falkner-Skan flows of conventional boundary layers. In a special case (magnetic-field strength constant) the problem is reduced to solution of the Blasius equation and a subsequent quadrature.

W. R. Sears (Ithaca, N.Y.)

6906:

Zhigulev, V. N. Theory of electrical discharge in a moving conducting medium. Soviet Physics. Dokl. **124** (4) (1959), 61–64 (1226–1228 Dokl. Akad. Nauk SSSR).

Consider an electrical discharge from a line electrode to a cylindrical electrode of semi-circular section and large radius. The line electrode lies along the central axis of the cylinder, so that in the two-dimensional problem seen in a transverse plane the discharge is from a point O to a large semicircle. The whole is immersed in an incompressible fluid of high electrical conductivity, and it is supposed that this fluid is made to flow uniformly in a parallel stream toward the (permeable) outer electrode.

The fluid motion tends to transport the discharge, so that the result is a narrow jet of electric current flowing from O substantially in the stream direction. This constitutes one of the magnetic-boundary-layer cases of the paper reviewed directly above, and the results can be carried over. The analogous axisymmetric case, the physical interpretation of which is presumably a discharge from a point electrode toward a hemispherical collector, is also treated. The temperature distributions are calculated.

The motivation of the investigation seems to be that, because the current flux is confined to a narrow region, very high temperatures may result.

W. R. Sears (Ithaca, N.Y.)

6907:

Greenspan, H. P.; and Carrier, G. F. The magnetohydrodynamic flow past a flat plate. J. Fluid Mech. **6** (1959), 77–96.

The problem considered is the steady flow of a viscous incompressible electrically conducting fluid past a semi-infinite or finite plate when the applied magnetic field is parallel to the plate. The main results are found using the modified Oseen approximation in which the convection terms are linearized by replacing the convective velocity by a suitable fraction of the main stream velocity. The linearized problem is then formulated in terms of integral equations which are solved by the Wiener-Hopf technique for the semi-infinite plate and an approximate method valid for small Reynolds number (short plate, say) for the finite plate.

This paper was probably the first one to draw attention to the fact, familiar now, that the propagation of Alfven waves upstream in sub-Alfven flow makes that case very different from the super-Alfven flow. For the semi-infinite plate there is no solution for the sub-Alfven case which leaves the original uniform conditions at infinity undisturbed. In the super-Alfven case the boundary layer gets thicker as the velocity decreases and the flow is eventually plugged as the critical velocity is reached. These results are checked by numerical integration of the non-linear boundary layer equations, as well as by a special analysis of the behavior near the critical velocity.

For the finite flat plate a flow is also found for the sub-Alfven case. The Alfven waves propagate upstream but also diffuse, for finite viscosity and conductivity, and do not violate the uniform conditions at infinity. The presumption is that the upstream influence (rather like a reversed wake) penetrates further as the plate length increases and in the limit of the semi-infinite plate the whole fluid moves with the plate. It should be noted, however, that the flow still plugs at the Alfven speed, and one would suspect the validity of the linear theory for that case.

G. B. Whitham (Cambridge, Mass.)

6908:

Korobeinikov, V. P.; and Ryazanov, E. V. On solutions to the equations of magnetogasdynamics with zero temperature gradient. Soviet Physics. Dokl. **124** (4) (1959), 40–42 (51–52 Dokl. Akad. Nauk SSSR).

It is supposed that heat exchange between particles is so strong that the temperature is uniform in the gas at every instant. The equations for cylindrical and plane wave propagation are written, neglecting viscosity and electrical resistance and assuming the magnetic field to be uniform in the plane case and axial, concentric, or helical in the cylindrical case. A particular solution is found and is used to write the jump conditions for a shock wave in terms of its propagation speed.

W. R. Sears (Ithaca, N.Y.)

6909:

Syrovatskii, S. I. The stability of shock waves in magnetohydrodynamics. Soviet Physics. JETP **35** (8) (1959), 1024–1027 (1466–1470 Ž. Èksper. Teoret. Fiz.).

"The interaction between shock waves in a magnetic field and magnetohydrodynamic waves of small amplitude is considered. The condition for stability with respect to spontaneous emission of weak magnetohydrodynamic waves by a shock wave has been obtained. The conditions are found under which the linear equations for a small perturbation do not have a solution. This case is interpreted as the decay of the shock wave." (Résumé de l'auteur)

J. Naze (Marseille)

6910:

Agostinelli, Cataldo. Su di una classe notevole di figure ellissoidali rotonde di masse fluide magnetoidrodinamiche uniformamente rotanti e gravitanti. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 369–381.

"On étudie une distribution stationnaire particulièrement intéressante du champ magnétique à l'intérieur et à

l'extérieur d'une masse fluide incompressible de conductivité électrique infinie, en rotation uniforme autour d'un axe barycentrique, sujetie à la gravitation propre, et que l'on suppose de forme ellipsoïdale." (Résumé de l'auteur) *J. Naze* (Marseille)

6911:

Riazanov, E. V. The solution of the equation of magnetohydrodynamics describing the one-dimensional axisymmetrical motion of a gas in a gravitational field. J. Appl. Math. Mech. **23** (1959), 260–265 (187–189 Prikl. Mat. Meh.).

A cylinder of gas subject to its self-gravitation and to the action of a magnetic field is considered. The motion is assumed to be such that the velocity is always perpendicular to the axis of the cylinder, varies linearly with the distance from the axis and is also proportional to a function μ of the time. The equations of magnetohydrodynamics are solved exactly. The various types of motions are briefly described; they differ from one another according to the nature of the function μ. The value of the magnetic field at the outer boundary of the cylinder and the sign of the pressure at points inside the cylinder are matters that are not expressly mentioned.

G. C. McVittie (Urbana, Ill.)

6912:

Ham, Frank S. Shape-preserving solutions of the time-dependent diffusion equation. Quart. Appl. Math. **17** (1959), 137–145.

The author obtains explicit solutions of the diffusion equations corresponding to the diffusion-limited growth of precipitate particles of ellipsoidal shape and of dimensions which grow proportional to the square root of time.

C. R. DePrima (Pasadena, Calif.)

6913:

Aris, R. On the dispersion of a solute by diffusion, convection and exchange between phases. Proc. Roy. Soc. London. Ser. A **252** (1959), 538–550.

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 6531, 6758, 6905, 6906, 6970, 7053, 7116.

6914:

Bartkowski, Zygmunt. Zur Berechnung des Körte-Spiegels. Z. Angew. Math. Mech. **39** (1959), 325–326.

6915:

Raman, C. V. Huyghens' principle and the diffraction of light. I. Theoretical considerations. Proc. Indian Acad. Sci. Sect. A **50** (1959), 83–90.

6916:

Jánossy, L. The fluctuations of intensity of an extended light source. Nuovo Cimento (10) **12** (1959), 369–384. (Italian summary)

The author extends calculations on the fluctuation of intensity of an extended light source, carried out previously by himself [Nuovo Cimento (10) **6** (1957), 111–124; MR **19**, 802] and shows that the theory is able to account for a number of observed experimental effects.

H. Messel (Sydney)

6917:

Duffieux, P. Michel. Les franges d'interférences de deux systèmes d'ondes et la théorie de l'information. Rev. Opt. **37** (1958), 441–457.

In this article the author discusses the problem of the degree of visibility of fringes arising from a two-beam system of waves originating from a point object or from a one-dimensional finite incoherent light source. This problem, as well as related cases dealing with multiple beams interference phenomena, is of great practical importance in obtaining information about the structure of objects, spectral lines, etc. Some well known optical systems which are included in this class of problems are the Fresnel biprism, Billet's split lens, the Michelson and other interferometric instruments and also the recently developed diffractometer.

Here the author discussed the localization of the interference pattern (sinusoidal form) produced by two beams in the common region of the image space (receiving screen), when diffraction effects due to the finite extent of the aperture (diaphragm) are neglected, as well as the visibility of the fringes about the zero order fringe. For an incoherent light source of finite extent, the degree of visibility depends linearly on the Fourier transform of the source. This is followed by a discussion of the effect of diffraction on the degree of information, which is obtained, on the basis of geometrical theory.

{Reviewer's remark: The theoretical analysis of such systems discussed in this paper, including more general optical systems and sources, are systematically treated by Born and Wolf in chapters VII and X of their excellent book *Principles of optics*, reviewed below.}.

N. Chako (Flushing, N.Y.)

6918:

★Born, Max; and Wolf, Emil. Principles of optics: Electromagnetic theory of propagation, interference and diffraction of light. Pergamon Press, New York-London-Paris-Los Angeles, 1959. xxvi+803 pp. (19 plates) $17.50.

This book is not at all an English version of the first author's famous "Optik" [Berlin, Springer, 1933]. Compared to "Optik", its scope is restricted to a narrower field. For example, classical molecular optics, the subject-matter of almost half of the German book, is not included. In fact, the new book is restricted to those parts of optics which can be treated in terms of Maxwell's phenomenological theory. Notwithstanding, the new book is larger than the old one by an odd 200 pages, giving an indication about the extent of the research that has been done in classical optics in recent times. Chapter headings and numbers of pages are as follows. I, Basic properties of the electromagnetic field (69); II, Electromagnetic potentials and polarization (38); III, Foundations of geometrical optics (24); IV, Geometrical theory of optical imaging (70); V, Geometrical theory of aberrations (30); VI, Image-forming instruments (23); VII, Elements of the theory of interference and interferometers (114); VIII, Elements of the theory of diffraction (89); IX, The diffraction theory of aberrations (32); X, Interference and

diffraction with partially coherent light (63); XI, Rigorous diffraction theory (37); XII, Diffraction of light by ultrasonic waves (18); XIII, Optics of metals (54); XIV, Optics of crystals (54). These chapters are preceded by an 8-page historical introduction and followed by 9 appendices, of 53 pages in total. There are extensive author and subject indexes. Several parts of the book are contributed by Bhatia, Clemmow, Gabor, Stokes, Taylor, Wayman, and Wilcock. This valuable book should and will find a place on the shelves of any worker in the field of optics.

C. J. Bouwkamp (Eindhoven)

6919:

Stepanov, K. N. Penetration of an electromagnetic field into a plasma. Ž. Èksper. Teoret. Fiz. **36** (1959), 1457–1460 (Russian); translated as Soviet Physics. JETP **9**, 1035–1036.

The depth of penetration of an electromagnetic field into a semi-infinite plasma in a magnetic field perpendicular to the plasma boundary is calculated.

Author's summary

6920:

Vedenov, A. A.; and Larkin, A. I. Equation of state of a plasma. Ž. Èksper. Teoret. Fiz. **36** (1959), 1133–1142 (Russian); translated as Soviet Physics. JETP **9**, 806–811.

Using a graphical method similar to the Feynman method in quantum electrodynamics the authors found an expression for the free energy F of a completely ionized gas in terms of an expansion in the density of the interacting particles n: $F=F_{\text{ideal}}+An^{3/2}+Bn^2\ln n+Cn^2$. The term $An^{3/2}$ is shown to be identical with the well-known Debye-Hückel term and a method is given to calculate the expressions for B and C. *R. S. B. Ong* (Ann Arbor, Mich.)

6921:

Adawi, I. Variational approach to deviations from Ohm's law. Phys. Rev. (2) **115** (1959), 1152–1156.

In a perturbation framework, Kohler's variational method has been extended to obtain deviations from Ohm's law for a nondegenerate electron gas. Solution for the distribution function reduces to solving sets of linear algebraic equations. It is shown that, to second order in the field strength, the "popular" Maxwellian distribution (with a new temperature) is only a first-order variational solution. The method becomes extremely simple if the diffusion approximation is introduced and a relaxation time can be defined. Under these conditions, the second-order term in the mobility is expressed as the ratio of two infinite determinants using the usual representation, in which the unknown function is expressed as an energy polynomial. This ratio can be expressed by an infinite series.

Author's summary

6922:

Huxley, L. G. H. The structure of a stream of electrons and ions drifting and diffusing in a gas when ionization by collision and molecular attachment are present. Austral. J. Phys. **12** (1959), 171–183.

6923:

Berkowitz, Jerome; and Gardner, Clifford S. On the asymptotic series expansion of the motion of a charged particle in slowly varying fields. Comm. Pure Appl. Math. **12** (1959), 501–512.

This paper gives the proof of convergence of an expansion by Kruskal [to appear in Nuovo Cimento].

W. P. Allis (Cambridge, Mass.)

6924:

Itoh, Makoto. The unified electromagnetic equation and its properties in curvilinear coordinate systems. Univ. Nac. Tucumán. Rev. Ser. A **12**, 85–106 (1959).

If one applies Mikusiński's operational calculus (which is formally equivalent to the Laplace transform) to the first two of Maxwell's equations,

$$\operatorname{curl}\mathbf{E}=-\mu\dot{\mathbf{H}},\quad \operatorname{curl}\mathbf{H}=\varepsilon\dot{\mathbf{E}}+\sigma\mathbf{E},$$

in the usual notation, one gets

$$\operatorname{curl}\mathbf{E}=-\mu p\mathbf{H}+\mu\mathbf{H}_0,\quad \operatorname{curl}\mathbf{H}=\varepsilon p\mathbf{E}+\sigma\mathbf{E}-\varepsilon\mathbf{E}_0.$$

These can be combined into

$$(*)\qquad \operatorname{curl}\mathbf{F}=\kappa(\mathbf{F}-\mathbf{F}_0),$$

where

$$\mathbf{F}=\mathbf{E}+\eta\mathbf{H},$$

$$\mathbf{F}_0=\frac{\varepsilon}{\varepsilon p+\sigma}\mathbf{E}_0-\frac{\mu}{\kappa}\mathbf{H}_0,$$

$$\eta=i\left(\frac{\mu p}{\varepsilon p+\sigma}\right)^{1/2},\quad \kappa=(\varepsilon p+\sigma)\eta.$$

(*) is the unified electromagnetic equation of the title. The idea of such a unified equation is not new: see, for example, H. Weber, Ann. Physik. (4) 8 (1902), 721–751; H. Bateman, *Electrical and optical wave-motion*, University Press, Cambridge, 1915; p. 16. The novelty here is its use with the operational calculus.

The author then determines, by the aid of the tensor calculus, the conditions under which a component of the unified electromagnetic vector $\mathbf{F}$ satisfies a second-order partial differential equation. There are two cases: (A) generalized cylindrical coordinates with metric

$$ds^2=\{g_1(x_1,x_2)\}^2dx_1^2+\{g_2(x_1,x_2)\}^2dx_2^2+dx_3^2$$

where

$$\frac{\partial}{\partial x_1}\left\{\frac{1}{g_1}\frac{\partial g_2}{\partial x_1}\right\}+\frac{\partial}{\partial x_2}\left\{\frac{1}{g_2}\frac{\partial g_1}{\partial x_2}\right\}=0;$$

(B) generalized spherical polar coordinates with metric

$$ds^2=x_3^2\{h_1(x_1,x_2)\}^2dx_1^2+x_3^2\{h_2(x_1,x_2)\}^2dx_2^2+dx_3^2$$

where

$$\frac{\partial}{\partial x_1}\left\{\frac{1}{h_1}\frac{\partial h_2}{\partial x_1}\right\}+\frac{\partial}{\partial x_2}\left\{\frac{1}{h_2}\frac{\partial h_1}{\partial x_2}\right\}=-h_1h_2.$$

Finally he describes how solutions may be obtained in these coordinate systems, but it is impossible to give details briefly. *E. T. Copson* (St. Andrews)

6925:

Barsukov, K. A. On the Doppler effect in an anisotropic and gyrotropic medium. Ž. Èksper. Teoret. Fiz. **36** (1959), 1485–1491 (Russian); translated as Soviet Physics. JETP **9**, 1052–1056.

The usual theory of the Doppler effect concerns only the case in which the source moves in free space or in a homogeneous medium. In certain problems, such as the motion of a rocket through the ionosphere, the medium in which

the emission and propagation of the radiation take place is both anisotropic and gyrotropic. The author investigates the theory of emission of radiation in such a medium, for moving oscillator sources. The treatment is formal in the sense that the medium is described mathematically by means of a phenomenological permeability tensor.

E. L. Hill (Minneapolis, Minn.)

6926:

Karp, Samuel N.; and Karal, Frank C., Jr. Vertex excited surface waves on both faces of a right-angled wedge. Comm. Pure Appl. Math. **12** (1959), 435–455.

Electromagnetic surface waves are produced by a magnetic line dipole source located at the corner of a right-angled wedge, with an impedance boundary condition of the form

$$\frac{1}{r}\frac{\partial u}{\partial \theta}+\lambda u = 0 \quad (\theta = 0),$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta}-\lambda u = 0 \quad (\theta = \alpha)$$

prescribed on the wedge surface. (Here u is the component of the magnetic vector parallel to the line of intersection of the plane faces of the wedge.) The constant λ is characteristic of the surface and is such that surface waves are generated. The complete field is found for wedges of angle $\alpha=\pi$ and $\alpha=\frac{1}{2}\pi$ by first solving a differential equation for a simpler auxiliary variable. This method was used earlier by Stoker and by Lewy in water wave theory.

F. Ursell (Cambridge, England)

6927:

Faĭn, V. M. On the theory of the coherent spontaneous emission. Ž. Èksper. Teoret. Fiz. **36** (1959), 798–802 (Russian); translated as Soviet Physics. JETP **9**, 562–565.

Recent experimental and theoretical studies have emphasized the importance of line-shifts and line-widths for electromagnetic radiation of all frequencies. The author deals with this problem in a simplified and very approximate way for a system of radiators which are close enough to be coupled together by their induction (as well as their radiation) fields. Each radiator is treated formally as a system which can occupy two quantized states. Formulas are developed for the line-shift produced by the coupling, both in classical and quantum mechanical theory. {In the approximation considered here the deeper questions of principle associated with the concept of transition probabilities are not treated.}

E. L. Hill (Minneapolis, Minn.)

6928:

Lowndes, J. S. An application of the Kontorovich-Lebedev transform. Proc. Edinburgh Math. Soc. **11** (1958/59), 135–137.

The author gives a second method of solving a problem treated in a recent paper, "The diffraction of transient electromagnetic waves by a wedge", by A. C. Butcher and J. S. Lowndes [same Proc. **11** (1958), 95–103; MR **20** #6277]. He solves Maxwell's equations formally, for the electric field in the space between perfectly conducting planes $\theta=0$ and $\theta=\alpha$ when an infinite line of time-variable current, parallel to the z axis, lies in that region. Iterated integral transformations are used here: the Laplace transformation with respect to time t and the transformation $\int_0^\infty f(r)K_\lambda(\eta r)r^{-1}dr$ with respect to r, where K_λ is a modified Bessel function.

R. V. Churchill (Ann Arbor, Mich.)

6929:

Zăgănescu, M.; et Toro, T. Sur l'utilisation de la fonction δ de Dirac dans l'éléctrodynamique classique. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 201–205 (1959). (Romanian. French and Russian summaries)

On introduit deux nouvelles fonctions δ_s, δ_l analogues à la fonction de Dirac et qui permettent d'écrire d'une manière correcte les équations de l'éléctrodynamique classique.

Par exemple, l'équation de Poisson s'écrit

$$\Delta\varphi = -4\pi(\rho+\delta_s\sigma),$$

σ étant la densité superficielle des charges éléctriques. Le potentiel vectoriel A satisfait à l'équation $\Delta A = 4\pi/c(\mathbf{j}+\delta_l I\mathbf{t})$, I étant l'intensité des courants linéaires.

Résumé des auteurs

6930:

Horváth, János. Eine Axiomatisierung der Maxwellschen Theorie des elektromagnetischen Feldes. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 259–286. (Hungarian)

A German version of this article [Acta Phys. Acad. Sci. Hungar. **8** (1958), 399–418] has already been reviewed [MR **20** #2979].

6931:

de Belatini, Paul. The electrodielectric induction. Bull. Tech. Univ. Istanbul **10** (1957), 84–111. (Turkish summary)

"In a former paper [same Bull. **6** (1953), 11–28] the author has shown a complete duality between magnetic and dielectric quantities and laws, without going into details of physical phenomena and devices characterised by these quantities and governed by these laws. The object of this paper is to fill this gap and extend the duality as far as possible. It will be shown that the dielectric domain, which is generally treated only as regards statical phenomena (electrostatics), has also its dynamical phenomena which are dual to those of dynamical magnetics. It will be shown also that an 'electrodielectric induction' exists, but for this purpose a new capacitive device must be introduced, which is the correct dual of a single turn or current loop and this device is not the usual capacitor." (Author's summary)

K. Maruhn (Giessen)

6932:

Eichhorn, H. Bemerkung zum Potential eines homogenen Rotationsellipsoides. Astr. Nachr. **283** (1957), 249–250.

Darstellung des Potentials eines homogenen Rotationsellipsoides in seinem Aussenbereich durch neue Koordinaten, die mit den elliptischen nahe verwandt sind.

K. Maruhn (Giessen)

6933:

★**Brillouin, Léon. Wave propagation and group velocity.** Pure and Applied Physics, Vol. 8. Academic Press, New York-London, 1960. xi+154 pp. $6.00.

This volume contains the following articles: "About the propagation of light in dispersive media" by A. Sommerfield, which is a translation of the paper in Ann. Physik (4) **44** (1914), 177–202; a paper of the same title by L. Brillouin, ibid., 203–240; "Propagation of electromagnetic waves in material media" and "Wave propagation in a dispersive dielectric" by L. Brillouin, which were first published in *Congrès International d'Électricité*, Paris, 1932, vol. 2, pp. 739–788 [Gauthier-Villars, Paris, 1933]. There is a new introduction placing these papers in their historical perspective and an added chapter on waves in wave guides and other examples.

6934:

Al'pert, Ya. L. Computation of the field of LF and VLF radiowaves over the Earth's surface under natural conditions. Radiotehn. i Elektron. **1** (1956), 281–292; available in translation **T320** by E. R. Hope. Directorate of Scientific Information Service, Defense Research Board, Ottawa, Canada, Nov. 1959.

The paper sets out to compute the time-harmonic electromagnetic fields in the frequency range 500 c/s to 30 kc/s at large distances along the surface of the earth, from a transmitting antenna. The adapted model is a parallel-plate waveguide whose lower boundary is the air-ground interface and whose upper boundary, at height h, is the lower edge of the ionosphere D region. The source is represented by a vertical electric dipole. The calculations are based on the general integral contour representation for the planar problem. This is transformed to a sum of modes which individually correspond to the residues of the poles of the integrand. The resulting pole-determining equation involves the reflection coefficient $\rho_2(C)$ of the upper boundary for plane waves incident at a (complex) angle whose cosine is C. The conductivity of the D region is allowed to vary with height $h+z$ according to the function $[1+\exp(-mz)]^{-1}$ where m is chosen to be 0.413 to correspond to best available published data (the influence of the earth's magnetic field is neglected). Taking P. S. Epstein's formula [Proc. Nat. Acad. Sci. U.S.A. **16** (1930), 627–637] for the reflection coefficient $\rho(C)$, the author then uses a semi-empirical technique described as the "alignment method", to estimate $\rho_2(C)$. This is the crucial step in the analysis and its implications are not adequately discussed in the paper. Actually, Epstein's expression for $\rho(C)$ is valid only for horizontal polarization (i.e., the electric vector is horizontal), whereas the required expression $\rho_2(C)$ for the pole equation corresponds to vertical polarization (i.e., the magnetic vector is horizontal). The "alignment" procedure amounts to using Epstein's results for horizontal polarization to define an equivalent sharply bounded ionosphere whose conductivity is, strictly speaking, both a function of C and the frequency. It is then assumed that this same "effective" conductivity may be inserted into the usual Fresnel reflection formula for vertical polarization to yield a numerical value for $\rho_2(C)$. In view of the radically different behavior of the Fresnel reflection coefficients for horizontal and vertical polarization at grazing angles (i.e., small C) the validity of this procedure is doubtful.

The author then claims that his results may be corrected for the sphericity of the earth by multiplying the fields by the factor $(\theta/\sin\theta)^{1/2}$ where θ is the central angle. The adequacy of this step is questioned since the earth curvature modifies the pole-determining equation and the attenuation of the dominant modes would be increased if this were taken into account. This fact would have been evident if the problem had been formulated in spherical geometry at the outset. Actually, an earth-flattening approximation for the pole-determining equation is only valid when the spherical wave functions are adequately represented by their second order or Debye approximation. Such is the case when $(ka/2)^{1/3}C \ll 1$, where k is the wave number and a is the radius of the earth. This appears to be violated for many of the numerical results given in the paper. In fact, if curvature is completely neglected, a further restriction (which could be obtained from geometrical optics) is that $C \ll (h/a)^{1/2} \approx 0.1$. This is also violated in many of the cases treated by the author.

It is possible that the reported agreement with experimental data, in the frequency range 20 to 30 kc/s, is partly fortuitous since the "alignment procedure", mentioned above, yields overly high attenuation of the modes, whereas the neglect of earth curvature leads to very low attenuation. *J. R. Wait* (Boulder, Colo.)

6935:

Mercier, R. P. The propagation of fading waves. Phil. Mag. (8) **4** (1959), 763–776.

A thin diffracting screen is considered, producing small random perturbations of the phase and amplitude of a plane wave passing through it. Applying Fresnel's diffraction formula, the statistical properties of the wave are calculated at all distances. A critical distance from the screen is found, identical with the Rayleigh distance for a telescope of aperture equal to the average size of the screen irregularities. Within this critical distance, the relative amounts of phase and amplitude fluctuation are substantially the same as at the screen; beyond this distance, the fluctuations become uncorrelated with the screen variations. Some comparisons are given with radio experiments at 16 kc/s frequency.

S. A. Bowhill (University Park, Pa.)

6936:

Budden, K. G. Effect of small irregularities on the constitutive relations for the ionosphere. J. Res. Nat. Bur. Standards Sect. D **63D** (1959), 135–149.

The effect of irregularities, small compared to the wavelength, on the constitutive properties of an ionized medium are investigated in terms of an equivalent polarization vector. For those who prefer to regard the medium as a conductor rather than a dielectric the modifications required are only formal, since the "Lorentz force" proper to a dielectric is neglected.

The case of irregularities in one dimension is first considered. It is found that even when collisional effects are negligible, the square of the refractive index is complex. The corresponding loss of energy from an electromagnetic wave is attributed to scattering. There also appears an additional term, analogous to the "Lorentz term".

The more difficult theory for three-dimensional irregularities is developed by means of successive approximations. The results indicate that small irregularities may play an important part in the propagation of very-low-frequency radio waves in the ionosphere. In particular, they may explain why "whistlers" are observed only on comparatively rare occasions. *K. C. Westfold* (Sydney)

6937:

Samson, A. M. Angular distribution of resonance radiation emerging from a plane-parallel layer. Inž.-Fiz. Ž. **1** (1958), no. 1, 65–73. (Russian)

Approximate expressions are obtained for the intensity of radiation emerging from a plane-parallel layer. These expressions are derived for different angles of incidence and for irradiation of the layer by diffuse radiation. The formulas permit one to determine the spectroscopic characteristics of the radiation with an accuracy sufficient for experimental investigations.

J. E. Rosenthal (Passaic, N.J.)

6938:

Engler, A.; Kaplon, M. F.; and Klarmann, J. Some considerations on the analysis of primary cosmic ray intensity experiments. Nuovo Cimento (10) **12** (1959), 310–326. (Italian summary)

Many experiments on the primary cosmic radiation are carried out at a finite atmospheric depth. The difficult problem must then be faced of deducing from the results the primary particle intensities incident on the top of the earth's atmosphere. The authors examine critically the procedures followed in carrying out such a deduction and discuss the validity of the assumptions which must be made.

H. Messel (Sydney)

6939:

Pimenov, Yu. V. The plane problem of diffraction of electromagnetic waves by an ideally conducting strip of finite width. Ž. Tehn. Fiz. **29** (1959), 597–603 (Russian); translated as Soviet Physics. Tech. Phys. **4**, 532–538.

The author investigates the diffraction of an electromagnetic wave by a perfectly conducting, thin strip for (strip width) × (wave number) small. The method has been described for the scalar case by Bouwkamp [Rep. Progr. Phys. **17** (1954), 35–100; MR **16**, 200].

A. E. Heins (Ann Arbor, Mich.)

6940:

Lapidus, I. Richard; and Pietenpol, Jerry L. Classical interaction of an electric charge with a magnetic monopole. Amer. J. Phys. **28** (1960), 17–18.

"The motion of a charged particle in the field of a magnetic monopole is calculated classically. The differential scattering cross section is obtained, and it is seen that in the limit of small interaction strength the cross section resembles the Rutherford cross section. A simple relativistic correction is discussed." *Authors' summary*

6941:

Namioka, T. Theory of the concave grating. III. Seya-Namioka monochromator. J. Opt. Soc. Amer. **49** (1959), 951–961.

[For part II, see same J. **49** (1959), 460–465; MR **21** #556b.] The author discusses the optical mounting of a concave grating from the standpoint of diffraction theory, evaluating resolving power, and image errors.

M. Herzberger (Rochester, N.Y.)

6942:

Levine, Harold. Diffraction by an infinite slit. J. Appl. Phys. **30** (1959), 1673–1682.

The two-dimensional problem of diffraction by a slit is formulated as an integral equation and solved by a Wiener-Hopf procedure. Particular interest is centered on the case of a wide slit at grazing incidence, where the classical solutions to this problem are difficult to evaluate. The result obtained for the transmission cross section shows good agreement with exact results for narrower slits at grazing incidence. Finally comparison is made with a variational calculation of the scattering cross section of a strip.

E. T. Kornhauser (Bristol)

6943:

Hochstadt, Harry. Some diffraction by convex bodies. Arch. Rational Mech. Anal. **3** (1959), 422–438.

Asymptotic solutions are developed for plane wave scattering by a circular cylinder, a sphere and a parabolic cylinder. The class of problems discussed are stated concisely in the following manner: u, a scalar, is to satisfy (1) $\Delta u+k^2u=0$, (2) $u+u_i$ or its normal derivative is to vanish on the surface of body, (3) u is to satisfy the Sommerfeld radiation condition at infinity. In the above, u_i is the incident (scalar) wave. The solution to this problem is assumed to be of the form $u=\int_S Gf(s)ds$, where G is the free-space Green's function and S is the boundary surface. $f(s)$ is a suitably chosen kernel which must be selected so that the boundary condition is also satisfied. This leads to an integral equation which can be solved quite readily when suitable expansion theorems exist. $f(s)$ is then determined in the form of a line integral or sum which can be rewritten as a contour integral. The scattered far-fields are then found by saddle-point type integration(s). Two of the points of stationary phase correspond to the reflected and diffracted wave, except on the caustic where they coalesce.

It appears that the series expansion for the fields in the case of the cylinder and sphere converge very poorly along the direction of the incoming plane wave in the shadow region. Incidentally, the second of the series in equation (7) should have $\cos \nu_m\phi$ replaced by $\cos \nu_m(\phi-\pi)$. While the results for the cylinder and sphere are not new, the method is quite economical and certainly is much more straightforward than the usual technique based on separation-of-variables and the Watson transformation. The results for the parabolic cylinder were given formally by S. O. Rice [Bell System Tech. J. **33** (1954), 417–504; MR **15**, 843], who also employed a different method.

J. R. Wait (Boulder, Colo.)

6944:

Nussenzveig, H. M. Solution of a diffraction problem. Philos. Trans. Roy. Soc. London. Ser. A **252** (1959), 1–51.

This diffraction problem considers the solution of the two-dimensional Helmholtz equation for a wave guide consisting of two parallel plates at $y=\pm a$, $x>0$, and an infinite plane mouth at $x=0$, $|y|>a$. In the region I inside the wave guide ($x>0$; $|y|<a$) the field is expanded into modes according to:

$$u_I(x,y)=\cos\left(\frac{s\pi}{a}y\right)\exp\left\{-ix\left(k^2-\frac{s^2\pi^2}{a^2}\right)^{1/2}\right\}+\sum_{n=0}^{\infty}a_n\cos\left(\frac{n\pi}{a}y\right)\exp\left\{ix\left(k^2-\frac{n^2\pi^2}{a^2}\right)^{1/2}\right\};$$

the integer s characterizes the primary incident mode

(supposed as a TM wave). In the adjacent open half space $x<0$ (region II) the solution is represented by a proper Fourier cosine integral with respect to the variable y (called "k-representation"). The boundary conditions at the walls are then accounted for. The additional conditions of continuity of u and du/dx in the interface of I and II lead to an infinite set of linear equations (2.14) for the unknown amplitudes a_n. Approximative solutions of the latter are amply discussed.

Part I concerns a wide wave guide ($K=ka\gg 1$). The integrals representing the coefficients of the linear equations are approximated accordingly. Their solution can be obtained with the aid of a perturbation method, starting from the limiting case of a single wave-guide wall, the other wall disappearing to infinity. The solution of the single-wedge problem then resulting is represented in a form simpler than that obtained by Reiche [Ann. Physik. **37** (1912), 131–156]. The corresponding approximation for large $K=ka$, the "independent wedges approximation", is particularly useful for modes not too near the critical range (defined by n values near the transition from the unattenuated travelling modes with $n<K/\pi$, to the attenuated evanescent modes with $n>K/\pi$). The interesting case of an incident critical mode leads to an almost perfect reflection of the latter near the mouth of the wave guide.

Part II concerns the other limiting case of a narrow wave guide. The edge singularities then play an important role. The amplitudes a_n can be expanded here into powers $K^n \log^m K$ ($m\leqq n$). Here, as in part I, several iteration methods are introduced in order to solve the basic infinite set of linear equations.

H. Bremmer (Eindhoven)

6945:

Fedorov, F. I. Reflection and refraction of light in biaxial crystals. Inž.-Fiz. Ž. **1** (1958), no. 1, 41–52. (Russian)

An invariant method is used to find the relations between the amplitudes of waves reflected and refracted at the surface of a transparent biaxial crystal. It is shown that this approach greatly simplifies all derivations and permits one to put the result in a more graphic form.

J. E. Rosenthal (Passaic, N.J.)

6946:

Gutman, A. Application of asymptotic integration of the wave equation to the solution of certain waveguide and resonator problems. Dokl. Akad. Nauk SSSR **125** (1959), 1252–1255 (Russian); translated as Soviet Physics. Dokl. **4**, 456–459.

6947:

Clay, C. S. Propagation of band-limited noise in a layered wave guide. J. Acoust. Soc. Amer. **31** (1959), 1473–1479.

6948:

Pokrovskii, V.; Ulinich, F.; and Savvinykh, S. Non-local reflection in waveguides of varying cross section. Soviet Physics. Dokl. **124** (4) (1959), 108–110 (304–306 Dokl. Akad. Nauk SSSR).

Reflection in a waveguide of non-uniform cross-section is discussed. Propagation is in the $\pm z$-direction, fields and structure are independent of x, and the waveguide boundary is specified by $y=\pm f(\alpha z)$, where α is a small parameter; $f(z)$ is an analytic function having as singularities only simple poles and having no poles or zeros on the real axis and approaches the limits $f_\pm$ as z approaches $\pm\infty$. The conditions $\alpha f_0\ll 1$ and $kf_0\sim 1$, where f_0 is "the" width of the waveguide and k is the radian wave number corresponding to the frequency of operation, are assumed. Results, consisting of zeroth and first order approximations for the reflection coefficient, are based essentially on three previous papers [Soviet Physics. Dokl. **120** (3) (1958), 580–583 (504–507 Dokl. Akad. Nauk SSSR); Soviet Physics. JEPT **34** (7) (1958), 879–882, 1119–1120 (1272–1277, 1629–1631 Ž. Eksper. Teoret. Fiz.)]. To this reviewer the paper is excessively elliptical. The method of approach is interesting and the results are presumptively useful.

D. M. Kerns (Boulder, Colo.)

6949:

Harary, Frank. On the line-group of two-terminal series-parallel networks. J. Math. Phys. **38** (1959/60), 112–118.

A two-terminal network is a connected graph N with two points u and v, called terminals, distinguished from the other points and from each other. N is series-parallel with respect to u and v if through each line of N there is at least one path from u to v not touching any junction twice, and no two of these paths pass through any line in opposite directions. The composition $N_1\circ N_2$ of two series-parallel networks N_1 and N_2 is constructed by replacing each line of N_1 by a copy of the entire network N_2, preserving the order in both networks. The line-group of a two-terminal network N is the group of permutations of the lines of N which induce an adjacency-preserving permutation of the points, and leave u and v fixed.

The main result in the paper is a necessary and sufficient condition that each permutation in the line-group of $N_1\circ N_2$ (where N_1 is series-parallel) shall consist of a permutation of the copies of N_2 in accordance with an element of the line-group of N_1, followed by a permutation of the lines in each copy of N_2 separately, according to elements of the line-group of N_2. The condition is that either N_1 has no pairs of lines with common end-points, or N_2 cannot be regarded as the parallel connection of two non-empty subnetworks.

N_1N_2 denotes the parallel connection, and N_1+N_2 denotes the series connection (in the prescribed order) of the networks N_1 and N_2. Various identities involving sum, product, and composition of networks are discussed.

The author also shows how to construct the group and the line-group of a network under electrical equivalence. Here series as well as parallel interchanges are admitted, and a simple formula results.

A. M. Duguid (Providence, R.I.)

6950:

Debry, Robert; et Huyberechts, Simone. Quelques aspects de la théorie des graphes. Cahiers Centre Études Rech. Oper. no. 2 (1959), 29–88.

This is an expository paper in two parts. The first part deals with the elements of graph theory and the four constants, "le nombre cyclomatique", "le nombre chromatique", "le nombre de stabilité interne", and "le nombre de stabilité externe". The second part is a useful and

detailed account of the theory of networks (a network is a directed graph with a capacity function defined on its arcs), especially of the algorithms which have been developed for various problems of flow in networks.

A. M. Duguid (Providence, R.I.)

6951:

Ivanov, A. V.; and Ermakov, V. S. Application of Laguerre polynomials to the solution of telegraph equations. Inž.-Fiz. Ž. **1** (1958), no. 1, 6–16. (Russian)

The inverse of the Laplace transform

$$e^{-p\xi}[(p+\alpha-\beta)/(p+\alpha+\beta)]^{\alpha}/p \quad (\xi, \alpha, \beta \geq 0)$$

is found in terms of Laguerre polynomials. This result is then used, for example, to solve the following problem:

$$-\partial u/\partial x = Ri + L\partial i/\partial t,$$
$$-\partial i/\partial x = Gu + C\partial u/\partial t, \qquad (0 \leq x \leq l;\ t, R, L, C, G > 0)$$

with $u(x, 0) = i(x, 0) = 0$, $u(0, t) = E$, $u(l, t) = R_l i(l, t) + L_l \partial i(l, t)/\partial t$.

N. D. Kazarinoff (Madison, Wis.)

6952:

Gorodeckiĭ, P. G. Parametric excitation of harmonic oscillations in a linear oscillating circuit whose parameters change with time in nonperiodic fashion. Ž. Tehn. Fiz. **29** (1959), 580–583 (Russian); translated as Soviet Physics. Tech. Phys. **4**, 517–520.

"This paper deals with the behavior of a linear oscillating circuit when its parameters are subjected to nonperiodic changes with the time. It is shown that it is possible for harmonic oscillations of the current in the circuit to be produced." *Author's summary*

6953:

★Holbrook, James G. Laplace transforms for electronic engineers. International Series of Monographs on Electronics and Instrumentation, Vol. 10. Pergamon Press, New York-London-Paris-Los Angeles, 1959. xiii + 259 pp. $10.00.

Despite the plethora of books on linear transforms applied to circuit theory published since Carson's classical text of the early twenties, the author's monograph definitely deserves an honorable niche, that is, from the point of the audience for which it is written. He succeeds in covering with outstanding clarity a great deal of selected familiar ground, certainly most useful to electronic engineers, while assuming only the meagerest background in both mathematics and circuit analysis. From a reader already somewhat familiar with the subject the author is apt to elicit both fascination at his ability of getting somewhere from almost nowhere and distaste at low-grade appeals, apologies for doing mathematics at all, and posing as intricate such trivialities as standard analysis of mutual inductance.

After coverage of complex numbers and the elements of complex variable analysis up to and including residues (no branching), the Laplace transform and its inverse are reached by the engineer's route: Fourier series to real to complex to one-sided Fourier to Laplace transform, by means of the convergence-forcing exponential factor. Numerous simple and trenchant examples light the pleasant way. The most useful general Laplace transform theorems including shift, initial and final value, differentiation and integration, and convolution are covered next; the exposition is excellent. There follows a necessarily meager, but unnecessarily uninspiring, chapter on selected topics of circuit analysis, almost exclusively steady state, and thus of questionable motivation in the context. The final chapters deal with transforms of special wave shapes and pulses and with specialized transform applications, including such diverse topics as series summation methods and maximally flat filters. Appendices offer driving and transfer function transforms, the latter including some active networks, and the usual tables of transforms of elementary functions.

Printing and illustrations are clear. Misprints are few and transparent, almost of educational value.

H. G. Baerwald (Albuquerque, N.M.)

6954:

Čegis, I. A.; and Yablonskiĭ, S. V. Logical methods of control of work of electric schemes. Trudy Mat. Inst. Steklov. **51** (1958), 270–360. (Russian)

Consider a two-terminal contact network N with admittance function $f(x_1, \cdots, x_n)$, x_i $(i = 1, \cdots, n)$ ranging over 0, 1. Let $f_i(x_1, \cdots, x_n)$ and $f_{\bar{i}}(x_1, \cdots, x_n)$ denote, respectively, the admittance functions of networks resulting from shorting and opening ith contact in N. Let $\mathfrak{R}$ be a set of unordered pairs (i, j), $i \neq j$, $i, j = 1, \cdots, n$. Then a set of n-tuples $T = \{(x_1, \cdots, x_n)\}$ is a "test" relative to $\mathfrak{R}$ if $f_a(x_1, \cdots, x_n) \not\equiv f_b(x_1, \cdots, x_n)$ over T for every pair (a, b) in $\mathfrak{R}$ (a and b can be primed or unprimed). Let e_k^{ij} be an n-tuple $(x_1, \cdots, x_n)$ such that $f_i(e_k^{ij}) \neq f_j(e_k^{ij})$ (i.e., e_k^{ij} is an input which distinguishes between f_i and f_j) and let E_{ij} be the set of all such n-tuples. A straightforward but very cumbersome way of obtaining T is to form the intersection $\bigcap_{(i,j)\in\mathfrak{R}} E_{ij}$ and express the result as a union of sets of n-tuples. Each such set will represent a minimal test, i.e., a set having the smallest number of elements. The authors give a number of artifices for simplifying the passage from expressions of the form $\bigcap\bigcup$ to $\bigcup\bigcap$ and, in addition, consider special structures of contact networks for which minimal tests can be obtained with relative ease.

L. A. Zadeh (Berkeley, Calif.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 6311, 6869, 6873, 6874, 6891.

6955:

Hofmann, Rudolf. Zur Lösung spezieller Wärmeleitungsprobleme. Z. Angew. Math. Phys. **10** (1959), 233–244. (English summary)

The author presents a manipulative procedure for solving some special boundary value problems in the heat equation for a finite cylinder, a sphere and a rectangular plate. He uses Laplace transforms in conjunction with other classical methods to obtain temperature formulas that involve complicated series. He does not show why his procedure for obtaining inverse Laplace transforms will give all terms of the inverse transform.

R. V. Churchill (Ann Arbor, Mich.)

6956:

Wilkins, J. Ernest, Jr. Conduction of heat in an insulated spherical shell with arbitrary spherically symmetric sources. SIAM Rev. **1** (1959), 149–153.

The author obtains a formula for the temperatures $T(r_1, \theta)$ at the inner surface of a spherical shell, when $T(r, \theta)$ satisfies the heat equation $rT_t = \alpha[rT_{rr} + 2T_r + f(r)]$ $(t > 0, r_1 < r < r_2)$, and the conditions $T(r, 0) = 0$, $T_r(r_1, \theta) = T_r(r_2, \theta) = 0$. His formula for temperatures $T(r_1, \theta)$ is an approximate one, in which time, θ, is assumed to be small, found with the aid of Laplace transforms. Some particular distributions $f(r)$ of sources are considered.

R. V. Churchill (Ann Arbor, Mich.)

6957:

Kudryšev, L. I.; and Ipatenko, A. Ya. Effect of free motion on the heat transfer coefficient for flow around a sphere in the region of small Reynolds numbers. Ž. Tehn. Fiz. **29** (1959), 309–318 (Russian); translated as Soviet Physics. Tech. Phys. **4**, 275–284.

"A theoretical solution for the problem of heat exchange by mixed convection is considered. The results of this solution are compared with experimental data. The calculated formula for the total coefficient of heat transfer by convection for flow around a sphere in the region of small Reynolds numbers is presented." *Authors' summary*

6958:

Jost, Wilhelm. Ein thermodynamisches Problem. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. II **1959**, 25–28.

6959:

v. Krzywoblocki, M. Z.; and Bergonz, F. H. On some problems of heat transfer in free molecule flow. Acta Phys. Austriaca **12** (1958/59), 400–411.

"The authors present briefly the general theory for aerodynamic heating in free molecule flow. This is then applied to a cone and a paraboloid of revolution in an attempt to predict surface temperatures. No effect of radiation to or from the body is considered." (From authors' summary) *D. W. Dunn* (Ottawa, Ont.)

6960:

Bak, Thor A. On the existence of oscillating chemical reactions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 116–129.

The possibility of oscillating chemical reactions has been postulated in a wide variety of problems, among them, biological reactions, electrochemical reactions, and growth phenomena. The mathematical analysis of oscillating chemical reactions leads to non-linear second-order differential equations, which are not amenable to ordinary iteration procedure, but are solved by a method due to Kryloff and Bogolubov. First order chemical reactions alone, in whatever mechanism, cannot lead to oscillating reactions. The author states the necessary and sufficient conditions for the occurrence of oscillating chemical reactions and stresses the lack of corroborative experimental work. *J. Ross* (Providence, R.I.)

QUANTUM MECHANICS

See also 6318, 6745, 6750, 6832, 6927, 7052.

6961:

★Landé, Alfred. Quantum theory from non-quantal postulates. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 353–364. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

This paper is substantially the same in content as an earlier one by this author [Phys. Rev. (2) **108** (1957), 891–893; MR **20** #6284]. *C. A. Hurst* (Adelaide)

6962:

★Järnefelt, G. On a finite approximation of the energy spectrum of the linear harmonic oscillator. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 118–134. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

After a brief review of the Heisenberg formulation of quantum mechanics for the linear harmonic oscillator, the author discusses an analogous formulation in which the infinite matrices of Heisenberg are replaced by finite matrices with coefficients in a finite field. Some explicit calculations of the eigenpolynomials have been made, including those for the prime 71 which required an electronic computer. *A. M. Gleason* (Cambridge, Mass.)

6963:

Fabri, E. Time reversal and complex numbers in quantum theory. Nuovo Cimento (10) **13** (1959), 326–343. (Italian summary)

The author shows in this paper how to go about reformulating quantum mechanics as a theory without complex numbers, and that the resulting theory has some advantages, particularly in discussing time reversal and related questions.

The imaginary unit of ordinary quantum mechanics is replaced by a two-dimensional matrix γ which satisfies $\gamma^2 = -1$ but has real elements. In this formalism the 'anti-linear' time-reversal operator χ can be given a matrix representation and satisfies the matrix equation $\chi\gamma + \gamma\chi = 0$. It is shown that χ^2 has the eigenvalue $+1$ when applied to a scalar wave-function, but -1 when applied to a spinor wave-function.

The possibility of attaching a physical meaning to time-reversed states is discussed. *H. S. Green* (Adelaide)

6964:

Sharp, R. T. Simple derivation of the Clebsch-Gordan coefficients. Amer. J. Phys. **28** (1960), 116–118.

"The three-dimensional angular momentum operator J takes a simple form when the operand is a scalar function $f(\xi, \eta)$ of the two complex variables ξ, η; so do the angular momentum eigenstates $f_m{}^j(\xi, \eta)$. Exploitation of this circumstance leads to a simple derivation of Racah's formula for the Clebsch-Gordan (vector addition) coefficients." *Author's summary*

6965:

Meckler, A. On the algebra of angular momentum. Nuovo Cimento (10) **12** (1959), supplemento, 1–40.

The Lie algebra of the 3-dimensional rotation group has received much attention in the literature of quantum mechanics as the algebra of angular momentum operators. Usually it is considered in its group theoretical aspects for the sake of its matrix representation theory. In the present paper the writer examines the abstract commutation algebra by itself, in association with the notation of tensors. The major problem which is studied is the construction of basis elements for the set of products of a given degree. This leads to the construction of special homogeneous polynomials known as multipole tensors. Some properties of these tensors are studied, and the connection with the usual representation theory is discussed.

E. L. Hill (Minneapolis, Minn.)

6966:

Garrido, L. M.; and Pascual, P. Diagonalization of hamiltonian. Nuovo Cimento (10) **12** (1959), 181–190. (Italian summary)

In the well known method of Foldy and Wouthuysen [Phys. Rev. (2) **78** (1950), 29–36] the Dirac hamiltonian $H=\alpha\cdot p+m\beta$ is diagonalized by a unitary transformation S: $H=S^{-1}HS=\beta(p^2+m^2)^{1/2}$. The present work extends this to particles of arbitrary spin, starting from the covariant wave equations $(\beta_\mu p_\mu+m)\psi=0$, and the associated hamiltonian form $(\alpha_k p_k+m\beta)\psi=p_0\psi$. The method is illustrated by applying it to the case of spin 0 and spin 1 particles. A table of corresponding operators in the new and the old representation is also given.

F. Villars (Cambridge, Mass.)

6967:

Ghosh, P. K. On the mathematical foundations of "physically observable functions". II. Bull. Calcutta Math. Soc. **50** (1958), 103–106.

A slight generalization and modification are given of a previous study of the present author, devoted to the "theory of physically observable functions" of Hosemann and Bagchi [Z. Physik **135** (1953), 50–84; MR **14**, 1072]. In contrast to the previous paper [Bull. Calcutta Math. Soc. **49** (1957), 25–28; MR **20** #4777] this work uses non-symmetric error functions and treats them as regularizing functions.

L. Van Hove (Utrecht)

6968:

Ciulli, S.; and Fischer, J. Partial wave analysis of the production of boson pairs. Nuovo Cimento (10) **12** (1959), 264–285. (Italian summary)

"Partial wave analysis for boson pair production on nucleons is made. The corresponding angular operators, which characterize the spin and angular dependence of the S-matrix, are expressed with the help of the Legendre polynomials and tabulated. Simultaneously a general method of calculating the angular operators is given for processes containing more than four particles, in which cases the straightforward method leads to very lengthy calculations." (Authors' summary)

P. W. Higgs (London)

6969:

Philippot, J. L'irréversibilité dans la théorie de la mesure. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 583–590. (English summary)

6970:

Phan-Van-Loc. Densité de source et principe de Huygens en théorie de l'électron de Dirac. C. R. Acad. Sci. Paris **249** (1959), 1467–1468.

6971:

Izyumov, Yu. A. Remarks concerning Holstein's and Primakoff's formalism. Dokl. Akad. Nauk SSSR **125** (1959), 1227–1230 (Russian); translated as Soviet Physics. Dokl. **4**, 402–405.

The concept of non-commuting operator algebras has been adopted by physicists as the basis of the theory of quantized fields. This procedure has led to a considerable literature concerning formal connections among systems of operators obeying various types of commutation relations. The present paper investigates such connections for the Clifford-Lie algebra of the rotation groups (spin algebra) and that of quantized Boson fields, in extension of earlier work by T. Holstein and H. Primakoff [Phys. Rev. (2) **58** (1940), 1098–1113]. The analysis is entirely formal, only finite-dimensional representations of the operators being considered.

E. L. Hill (Minneapolis, Minn.)

6972:

Young, Robert C. Conversion electron angular correlations: general K-shell formulation and threshold limit. Phys. Rev. (2) **115** (1959), 577–585.

The problem of the angular correlation of conversion electrons has been discussed many times before. The present paper gives some numerical calculations for the b_1 and b_2 coefficients as functions of Z for the polarization-independent, transversely polarized, and longitudinally polarized cases. A useful appendix gives a summary of the properties of relativistic Coulomb wave functions and claims to present these conveniently and consistently. If this is true, it would set a precedent in the literature. Another appendix gives some useful identities involving Racah coefficients.

M. J. Moravcsik (Livermore, Calif.)

6973:

Ciulli, Sorin; et Fischer, Jan. Sur la théorie des opérateurs angulaires. C. R. Acad. Sci. Paris **249** (1959), 1090–1092.

6974:

Gilson, J. G. Dispersion relation for nonlinear vacuum polarization effects. Phys. Rev. (2) **115** (1959), 752–754.

It is shown that if the vacuum polarization effects are treated beyond the first order, the current introduced in the vacuum by an external field is a nonlinear functional of the latter. Nevertheless, with suitable definition, the current can satisfy causality and an exact dispersion relation can be derived. The author suggests that it might be true generally that linearity is not necessary in discussing the equivalence of causality and dispersion relations.

M. J. Moravcsik (Livermore, Calif.)

6975:

Dyatlov, I. T. Dispersion relations for the electromagnetic form factor of the pion. Soviet Physics. JETP **36 (9)** (1959), 351–352 (505–507 Ž. Èksper. Teoret. Fiz.).

The dispersion relations for the electromagnetic form factor of a charged pion are written down; this form factor is related to the amplitude of π-π scattering in an approximation in which all but two-pion states are neglected. *E. C. G. Sudarshan* (Rochester, N.Y.)

6976:

Ishida, Kin-ichi. A note on the dispersion-theoretic approach to the one additional pion production by pion-nucleon collision. Progr. Theoret. Phys. **22** (1959), 499–512.

The dispersion relations for pion production in pion-nucleon scattering are derived using the scattering matrix element expressed by the application of the double Heisenberg operators fixing three of the five four-momenta relevant to the problem. No numerical calculations are given at the present time.

M. J. Moravcsik (Livermore, Calif.)

6977:

Slavnov, D. A.; and Sukhanov, A. D. On causality in a theory with an indefinite metric. Ž. Èksper. Teoret. Fiz. **36** (1959), 1472–1479 (Russian); translated as Soviet Physics. JETP **9**, 1044–1048.

The possibility of constructing a unitary and macroscopically causal scattering matrix in a theory with an indefinite metric is examined. The construction is carried out in the framework of perturbation theory by means of the interaction Lagrangians of the complete (physical plus sum of nonphysical) fields. By a special choice of the spectrum of the nonphysical fields it is possible to satisfy the requirements of unitarity and macroscopic causality in the second and third orders. These requirements cannot, however, be fulfilled together in the fourth order; thus within the framework of our assumptions it is not possible to construct a unitary and macroscopically causal scattering matrix in a theory with indefinite metric.

Authors' summary

6978:

Brenig, W.; und Haag, R. Allgemeine Quantentheorie der Stossprozesse. Fortschr. Physik **7** (1959), 183–242.

This article provides a very readable and up-to-date review of the general quantum theory of collisions. The topics discussed include on the one hand such mathematical problems as strong and weak convergence in a Hilbert space and their relation to the asymptotic conditions in quantum field theory [cf. R. Haag, Phys. Rev. (2) **112** (1958), 669–673; MR **20** #6296]; on the other hand more physical topics such as the relation between cross-sections and the S-matrix, the effective range theory and resonance theory are also developed. The last section deals with the analytical properties of the S-matrix and elementary dispersion relations. *P. W. Higgs* (London)

6979:

Greenberg, O. W. Haag's theorem and clothed operators. Phys. Rev. (2) **115** (1959), 706–710.

A proof is given of a result, the essential physical idea of which was first stated by Haag [Danske Vid. Selsk. Mat.-Fys. Medd. **31** (1957), no. 5], which shows that under rather general conditions a quantum field theory which is unitarily equivalent to a free field theory at a fixed time is equivalent to it for all time. As a consequence certain clothed operator theories are shown to be equivalent to free-field theories. Nevertheless, the authors hope that by relaxing the hypotheses of Haag's theorem, non-trivial forms of clothed theory may be obtained.

A. J. Coleman (Kingston, Ont.)

6980:

Cormack, A. M. Semiclassical theory of internal Rayleigh scattering. Phys. Rev. (2) **115** (1959), 619–623.

The effect of the atomic electrons on the angular distribution of nuclear multipole radiation is investigated. Approximating the situation by a spherically symmetric distribution of free electrons, the only effect is a phase shift in the emitted radiation field. This shift is estimated on the basis of a Thomas-Fermi approximation; from its smallness it is inferred that the effect of a non-spherical electron distribution on the angular pattern of nuclear radiation is too small to be detected, the corrections in amplitude being less than 1%.

F. Villars (Cambridge, Mass.)

6981:

Kacser, C. Higher Born approximations in non-relativistic Coulomb scattering. Nuovo Cimento (10) **13** (1959), 303–318. (Italian summary)

An attempt to calculate the differential cross-section for scattering in a Coulomb field, by the Born approximation, gives higher order terms which are infinite. This paper calculates the result for a Yukawa potential, for the limiting case of zero screening: the first three approximations are calculated exactly and show that the correct result for a Coulomb potential is reached with an infinite phase factor of unit modulus. *D. F. Mayers* (Oxford)

6982:

Nigam, B. P.; Sundaresan, M. K.; and Wu, Ta-You. Theory of multiple scattering: second Born approximation and corrections to Molière's work. Phys. Rev. (2) **115** (1959), 491–502.

A formula of Molière [Z. Naturforschung **2a** (1947), 133–145], widely used in the theory of multiple scattering of fast charged particles by atoms, is shown to be incorrect. Using a result of Dalitz [Proc. Roy. Soc. London Ser. A **206** (1951), 509–520] the authors obtain a new formula which they claim is correct to the second Born approximation. The two formulas are compared with experimental results for the scattering of 15.6 MEV electrons by Au and Be. *A. J. Coleman* (Kingston, Ont.)

6983:

Toptygin, I. N. Multiple scattering of polarized electrons. Soviet Physics. JETP **36 (9)** (1959), 340–346 (488–498 Ž. Èksper. Teoret. Fiz.).

The elastic scattering matrix for a spin $\frac{1}{2}$ particle in a centrally symmetric field contains two independent functions of the scattering angle. The scattering of

polarized particles can be expressed in terms of this and the equation of motion for the distribution function and the polarization vector can be written down; they are here solved approximately. "A solution which is valid for both small and large scattering angles has been obtained as a series expansion in spherical harmonics and spherical vectors." *E. C. G. Sudarshan* (Rochester, N.Y.)

6984:

Mitskevich, N. V. A nonlinear vacuum effect in gravitation theory. Ž. Eksper. Teoret. Fiz. **36** (1959), 1207–1211 (Russian); translated as Soviet Physics. JETP **9**, 859–861.

Second-quantization theory leads to an interaction between gravitons through the virtual quanta of other fields. For the case of a scalar field in a slowly changing metric a vacuum cosmological term is obtained by Schwinger's method. In contrast with electrodynamics there is a strong divergence; it is impossible to cut off the integral correctly, and this makes the conclusions only qualitative. The additional scattering of gravitons by a Schwarzschild field in a scalar particle vacuum is calculated. For low-energy gravitons, the effect is comparable with the nonlinear effect in the classical theory.

C. W. Kilmister (London)

6985:

Maksimov, L. A. Remarks regarding Heisenberg's paper on the Lee model. Soviet Physics. JETP **36** (**9**) (1959), 97–99 (140–144 Ž. Eksper. Teoret. Fiz.).

In the well-known field theoretical model originally proposed by Lee and afterwards investigated by several authors one has the trouble of anomalous states (ghost states) and negative probabilities. It has been proposed by Heisenberg [Nuclear Phys. **4** (1957), 532–563] that in spite of these difficulties one can give a physical interpretation of the Lee model in the particular case when the energy of the ghost state coincides with the energy of a normal state (a dipole ghost). To be able to obtain a unitary S-matrix in this case, i.e., a conservation of probability in a scattering experiment, Heisenberg found it necessary to use states where he had an admixture of an incoming "ghost wave" together with the real scattered particle. By choosing a suitable weight for this admixture, Heisenberg obtained a unitary S-matrix at least for some simple cases. So far it has been generally believed that this prescription works only for the dipole case (or possibly also in the case when the ghost has a complex energy). The foundation for this belief has been the observation that in the dipole case (as in the complex case) the norm of the ghost state is necessarily zero. Therefore, the admixture of incoming waves used by Heisenberg carries no probability and a suitable arrangement as indicated above gives probability conservation for the physical waves. In the paper to be reviewed here the author discusses the same problem for the case of a real energy of the ghost state. He suggests that also in this case one should allow an admixture of incoming ghost waves in every physical state. By choosing a suitable weight for this admixture he shows that it is indeed possible to get a kind of probability conservation for the physical states with positive norm. The procedure is perhaps best illustrated with the aid of the following simple example. Suppose that the incoming wave contains only one state with positive norm and one state with negative norm and that the same is true for the scattered wave. Schematically, we can then write

$$S\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}A\\b\end{pmatrix};\quad S\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}a\\B\end{pmatrix}.$$

In this notation a wave $\begin{pmatrix}x\\y\end{pmatrix}$ indicates a superposition of the state with positive norm with amplitude x and the state with negative norm with amplitude y. The symbol S is the S-matrix of the theory. The fact that the norm of a state is conserved in the scattering implies $|A|^2-|b|^2=1$ and $|a|^2-|B|^2=-1$. If the incoming wave is a mixture of states with positive and negative norm one finds

$$S\left[\begin{pmatrix}1\\0\end{pmatrix}+\xi\begin{pmatrix}0\\1\end{pmatrix}\right]=\begin{pmatrix}A+\xi a\\b+\xi B\end{pmatrix},$$

where $|A+\xi a|^2-|b+\xi B|^2=1-|\xi|^2$. The author suggests that if one chooses $|\xi|^2=|b+\xi B|^2$ the equation above can be interpreted to mean that probability is conserved for the states with positive norm involved in the scattering experiment. This scheme is carried through in some detail for certain simple subspaces of the Hilbert space of the Lee model. A similar scheme has also been proposed by B. Ferretti [see #6896]. *G. Källén* (Lund)

6986:

Ferretti, B. On quantization with an indefinite metric and the Lee model. Nuovo Cimento (10) **12** (1959), 393–430. (Italian summary)

This paper contains a discussion of the conventional Lee model in the case when the coupling constant is larger than the critical value and the existence of anomalous states is expected [G. Källén and W. Pauli, Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7; MR **17**, 927]. The author suggests a certain reinterpretation of the formalism for the purpose of getting a physically reasonable S-matrix in spite of the anomalous state in the theory. The argument goes the following way. It is first remarked that one can consider special cases of the parameters of the model where energy conservation forbids transitions between normal states and anomalous states. Such a case is obtained, e.g., if the cut-off function is different from zero only in a small energy interval which does not contain the energy difference between the normal and the anomalous V-particle states. Another and perhaps more interesting case occurs when the coupling constant g is above the critical value g_c but rather close to it. In that case the energy of the anomalous state is practically $-\infty$ and no transition to it can occur in a theory with a finite cut-off energy. Therefore, the S-matrix is unitary in these cases. The author then remarks that even in the more general case when g is much larger than g_c one can always write the interaction Hamiltonian as a sum of two terms, one of which contains a coupling constant g which corresponds to a unitary S-matrix according to the remark above and the other of which is an interaction energy which is roughly proportional to the difference Δg between the actual coupling constant and the one appearing in the first term. The author then introduces the two operators $U=\exp(-iHt)$ and $U_1=\exp(-i(H_0+H_1))$ where H_0 is the free particle Hamiltonian and H_1 is the first term in

H_{int} mentioned above. The S-matrix of the model is then defined to be

$$S = \lim_{\substack{t'\to\infty \\ t\to-\infty}} U_1^{-1}(t')U(t'-t)U_1(t),$$

and it is shown that this S exists and is unitary. With the aid of a somewhat intricate construction the author then makes a one-to-one correspondence between the states of this truncated theory and the states in the complete model which have positive norms. It is freely admitted in the paper that this whole construction is rather arbitrary, but it shows in the author's opinion a possibility of reinterpreting the formalism in a case which has so far been considered as "hopeless". *G. Källén* (Lund)

6987:

Hiida, Kichiro; Nakanishi, Noboru; Nogami, Yukihisa; and Uehara, Masayuki. Electromagnetic structure of the nucleon. I. Progr. Theoret. Phys. **22** (1959), 247–273.

The dispersion relations for the nucleon form factors are expected to have most important contributions from two and three pion intermediate states. In this paper the lowest order perturbation diagrams for these contributions are considered. In particular, the three pion diagram gives a mean square radius of the isoscalar magnetic moment distribution which is satisfactorily large (twice the experimental value), though the static moment of this distribution is too large by three orders of magnitude. The mean square radius of the isoscalar charge distribution has the correct order of magnitude, and changes sign at the nucleon Compton wavelength.

John G. Taylor (Cambridge, England)

6988:

Hiida, Kichiro; Nakanishi, Noboru; Nogami, Yukihisa; and Uehara, Masayuki. Electromagnetic structure of the nucleon. II. Progr. Theoret. Phys. **22** (1959), 351–372.

This paper gives a detailed account of the approximate evaluation of the perturbation diagrams used in the paper reviewed above. *John G. Taylor* (Cambridge, England)

6989:

Kaschluhn, F. On the asymptotic and causality conditions in quantum field theory. Nuovo Cimento (10) **12** (1959), 541–552. (Italian summary)

The main conclusion of the author is that the asymptotic condition as put forward by Lehmann, Symanzik and Zimmermann [Nuovo Cimento (10) **6** (1957), 319–333; MR **19**, 1133] is consistent only for a local field theory, i.e., for a theory where the field operators commute for space-like separations. As far as the reviewer can find, the main point in the argument is the observation that the application of this asymptotic condition leads to various formulae for scattering amplitudes, etc., where one has integrals such as

$$\int dy\, e^{iqy}\theta(x-y)[j(x), j(y)].$$

Here $j(x)$ is the current operator of the theory, $\theta(x)$ is the conventional step function for the time component of the vector x, and q is an energy momentum vector on the mass shell. The expression given by the integral above has, for various reasons, to be a scalar quantity. A sufficient condition for this is that the function under the integral sign is also a scalar quantity, i.e., that the two currents $j(x)$ commute for space-like separations. However, it is not clear to the reviewer whether this condition is also necessary. It rather appears that it should be possible to construct examples of functions which are non-invariant but have a support in p-space which lies entirely outside the mass shell. Any such function could be added to the expression multiplying the exponential above without changing the value of the integral. *G. Källén* (Lund)

6990:

Falk, David S. Green's function approximation method. I. The nucleon. Phys. Rev. (2) **115** (1959), 1069–1073.

The purpose of the paper is to construct an approximate Green's function for the nucleon which has the same analytic properties as the exact one. The approximation is in the assumption of non-correlation. The integral equation for the Green's function is symmetric in the two space-time points, and this is explained to be important. The linear integral equation in the approximation is completely renormalized. The equation is solved exactly and the asymptotic solution for large values of the mass parameter is investigated. *M. J. Moravcsik* (Livermore, Calif.)

6991:

Falk, David S. Green's function approximation method. II. The polaron. Phys. Rev. (2) **115** (1959), 1074–1078.

This is an application of the method discussed in the previous paper to the polaron. The same approximation is used and the same method is applied to solve the equation. A variational method is used to calculate the lowest energy state of the system. In spite of the crudeness in the detailed approximations used, the results agree with those of Feynman [same Rev. (2) **97** (1955), 660–665] for values of the "coupling constant" that are less than 3.

M. J. Moravcsik (Livermore, Calif.)

6992:

Misra, S. P. Application of Schwinger's action principle to quantise a fourth order meson field. Indian J. Phys. **33** (1959), 520–530.

We have here applied Schwinger's Action Principle to the case of a fourth order meson equation proposed by Bhabha and Thirring. The result obtained thus is not new, but the method illustrates with the simplest model the difficulties of applying Action Principle when the Lagrangian contains even the second order derivatives of the field operator, and gives a concrete and complete example of the generalisation of the Action Principle when the Lagrangian contains higher order derivatives as given by the author in a recent paper. *Author's summary*

6993:

Aramaki, Seiya. Dispersion relations and high energy limits in quantum field theory. Progr. Theoret. Phys. **22** (1959), 485–491.

The high energy behavior of field quantities is a central problem because it is connected with the consistency problem of present-day field theories. In the present paper this high energy behavior is studied starting from the

dispersion relations of Bogoliubov [Bogoliubov, Medvedev, and Polivanov, lectures on "Problems of the Theory of Dispersion Relations", unpublished] and of Symanzik [Phys. Rev. (2) **105** (1957), 743–749]. The method is based on the assumption of uniqueness of the solutions of dispersion relations. General remarks are made about the possibility of excluding zeros in the complex energy plane of forward reactions amplitudes.

M. J. Moravcsik (Livermore, Calif.)

6994:

★Ахиезер, А. И.; и Берестецкий, В. Б. Квантовая электродинамика. [Ahiezer, A. I.; and Berestetckiĭ, V. B. Quantum electrodynamics.] 2nd ed., revised. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 656 pp. 26.40 rubles.

The first (1953) edition of this book was reviewed in MR **16**, 431. The second edition has been completely reworked and as a result is 200 pages longer and even more useful than it was before. Notable additions: a chapter on the electrodynamics of spin zero particles and a treatment of polarized spin one-half particles.

A. S. Wightman (Princeton, N.J.)

6995:

Grawert, G.; Lüders, G.; and Rollnik, H. The *TCP* theorem and its applications. Fortschr. Physik **7** (1959), 291–328.

One of the more remarkable consequences of local quantum field theory is that invariance under the continuous proper Lorentz transformations implies invariance under the product of the discrete transformations of time reversal (T), charge conjugation (C), and space reflection (P). This theorem is of comparatively recent discovery, being slowly brought to light in greater and greater generality. It assumed great importance in elementary particle physics with the discovery that parity is not conserved in weak interactions. This review article contains an account of the proofs of the theorem together with its applications clearly presented in somewhat Germanic English.

J. C. Polkinghorne (Cambridge, England)

6996:

Zimmermann, W. One particle singularities of Green functions in quantum field theory. Nuovo Cimento (10) **13** (1959), 503–521. (Italian summary)

Scattering amplitudes appear to possess certain simple singularities which can be thought of as arising from the appearance of the vacuum or one particle states as real intermediate states. Attempted continuation to find the residues at these singularities is an important procedure in the analysis of experimental data. However the existence of these singularities has only been demonstrated in perturbation theory. In this paper it is shown by general methods that they do indeed have the expected form, at least in the case where the external momenta are all real four-vectors. Unfortunately this is not the case of greatest physical interest. *J. C. Polkinghorne* (Cambridge, England)

6997:

Mohan, G. Re-arrangement collision matrix. Nuovo Cimento (10) **13** (1959), 1065–1073. (Italian summary)

The collision matrix and state vectors are constructed for a two channel multiple particle system. It is shown, heuristically, how to generate a complete set of orthogonal states with given asymptotic properties despite the dual description physically necessary and where the asymptotic states are non-orthogonal. The state vector is separated by projection operators so that explicit reference to the x-representation is unnecessary.

E. A. Power (Manchester)

6998:

Ruelle, D. Analyticity of Wightman functions at completely space-like points. Helv. Phys. Acta **32** (1959), 135–137.

This note proves that in relativistic quantum field theory a vacuum expectation value

$$\langle A(x_0)A(x_1)\cdots A(x_n)\rangle_0 \tag{1}$$

of a local scalar field A is analytic at all completely space-like points, i.e., all points where $(x_j-x_k)^2<0$ for all $j, k=0, 1, \cdots, n$. Previous work had established that (1) is the boundary value of an analytic function

$$W(z_0, z_1, \cdots, z_n) = W(\zeta_1\cdots\zeta_n) \tag{2}$$

with $\zeta_k=z_k-z_{k-1}=\xi_k+i\eta_k$ and $z_k=x_k+iy_k$, where x_k and y_k and ξ_k and η_k are real four vectors. The domain of analyticity was known to include R_n', the extended tube (defined as all points $\Lambda\zeta_1, \Lambda\zeta_2, \cdots, \Lambda\zeta_n$ where Λ is in the proper complex Lorentz group, $\zeta_1, \zeta_2, \cdots, \zeta_n \in R_n$, and the tube R_n is the set of all $\zeta_1, \zeta_2, \cdots, \zeta_n$ such that η_k lie inside the aft-light cone). Furthermore, it was known to include all points obtainable from these by permuting the vectors z_j, i.e., at all points of $\bigcup PR_n'$, the union of the permuted extended tubes. The real points of R_n' (J-points) were determined by Jost. They are all completely space-like but have to satisfy the additional condition that $(\sum \lambda_j\zeta_j)^2<0$ for $\sum \lambda_j=1$, $\lambda_j\geq 0$.

Steinmann discovered that for $n\geq 3$ there are totally space-like points (S-points) which are not J-points and which do not become J-points under permutation of $x_0, \cdots, x_n$. The present note establishes that all S-points lie in the holomorphy envelope of $\bigcup PR_n'$. The proof works with complex times z_j^0 and real space components. Singularities can only occur at such points when two of the imaginary parts of the times become equal. To show that no singularity can occur when only one such pair of times is equal, one makes a complex Lorentz transformation to get the point into the tube. To go from m times equal to $m+1$, one uses the Kantensatz. The proof incidentally shows that $W(z_0, \cdots, z_n)$ is analytic at all points for which the times z_j^0, $j=0, \cdots, n$, are pure imaginary and the space components real provided no two times are equal.

A. S. Wightman (Princeton, N.J.)

6999:

Güttinger, W. Non-local structure of field theories with non-renormalizable interaction. Nuovo Cimento (10) **10** (1958), 1–36. (Italian summary)

The author examines the mathematical structure of pseudovector couplings of pseudoscalar mesons in quantum field theory. Such theories are unrenormalizable in the conventional perturbation sense due to the appearance of products of singular functions in higher terms of the perturbation series. The fundamental quantities in such a theory are transcendental functions of the conventional propagators, e.g., the exponential of the Δ_F function. The author uses recent results in generalized distribution

theory to give valid mathematical meaning to such structures which arose in the earlier work of R. Arnowitt and S. Deser [Phys. Rev. (2) **100** (1955), 349–361; MR **17**, 926] and L. N. Cooper [ibid., 362–370; MR **17**, 926]. The support of these functions is no longer a single point (i.e. the light cone) but becomes a set of points in the vicinity of the light cone. Thus the theory thereby acquires a non-local aspect (which is different from that of the standard non-local field theories with structure functions). The size of the non-locality is of the order of the coupling constant, which has dimensions of a length in this theory, while the nucleon's mass is the measure of the strength of the interactions at small distances. Estimates are given of the behavior of the generalized propagators at high frequencies: they go as complex exponentials of the frequency and hence the strength of the interaction effectively increases at very high energies.

R. Arnowitt (Syracuse, N.Y.) *and S. Deser* (Waltham, Mass.)

7000:

Kohn, W. Analytic properties of Bloch waves and Wannier functions. Phys. Rev. (2) **115** (1959), 809–821.

Bloch waves and Wannier functions for a one-dimensional symmetric periodic potential are considered as a function of complex wave-number. The most localized Wannier function is constructed, and its degree of localization is shown to be related to the width of the strip of analyticity of the corresponding Bloch functions in wave-number space. The domain of analyticity is shown to be limited by a set of branch points, and a procedure is given for locating these. *H. W. Lewis* (Madison, Wis.)

7001:

McIrvine, E. C.; and Overhauser, A. W. New quantum-mechanical representation. Phys. Rev. (2) **115** (1959), 1531–1536.

The authors make an interesting contribution to the problem of construction of orthonormal basis functions for a single particle in a periodic potential lattice. The Bloch functions, which are the closest approach to a momentum space representation, do not localize the particle in ordinary space. Wannier [same Rev. **52** (1937), 191–197] has used linear combinations of Bloch functions to construct a basis of functions giving localization of the particle in space, but not in momentum. The construction given here is an intermediate stage between these two methods, leading to a basis which provides some localization of both position and momentum. For this purpose a second lattice (superlattice) is introduced having no connection with the actual crystal lattice, and which may be given an arbitrary lattice spacing. The Bloch wave functions of the original lattice can be re-formed to represent Bloch functions for the superlattice, with a residue depending on the lack of conformity between the superlattice and the real crystal lattice. Linear combinations are then taken to provide the new basis of superlattice functions. Basis functions constructed in this manner may be useful for many applications. {The mathematical foundation for these procedures seems to rest on a general form of Wiener's theorem on the closure of translations of a function [*The Fourier integral*, University Press, Cambridge, 1933; § 15] here modified to take account of the periodicity of the crystal lattice}.

E. L. Hill (Minneapolis, Minn.)

7002:

Kazes, E. Generalized current conservation and low energy limit of photon interactions. Nuovo Cimento (10) **13** (1959), 1226–1239. (Italian summary)

Current conservation is used to demonstrate the relation between the low energy scattering of photons from charged particles and the static properties of the particles. The Compton scattering and bremsstrahlung amplitudes are written down to first order in the photon frequency. A similar expression is obtained for photo-pion emission, but this contains an arbitrary term.

D. J. Thouless (Birmingham)

7003:

Fried, Herbert M. Structure theorem for the photon propagator. Phys. Rev. (2) **115** (1959), 220–222.

A formal proof is given of the theorem whose physical contents are that the gauge-variant portion of the bare photon propagator is unchanged by the fermion-photon interaction; and that the gauge-variant portion of the dressed propagator must also be renormalized if the dressed photon operators are to be proportional to the renormalized photon operators.

M. J. Moravcsik (Livermore, Calif.)

7004:

Weinberg, Steven. Interference effects in leptonic decays. Phys. Rev. (2) **115** (1959), 481–484.

The paper deals with the qualitative difference between the "pure" V and A terms on the one hand, and the $V \cdot A$ interference terms on the other, in the decay of leptons. It is shown that: (1) if the lepton masses and charges are neglected, and no pseudoscalar correlations are measured, the "pure" terms are symmetric, while the interference terms are antisymmetric; (2) if the experiment measures pseudoscalar correlation, the conclusions are reversed; (3) if lepton charges and masses are included, but lepton spins and momenta are not observed, still only "pure" terms appear in scalars, and only interference terms in pseudoscalars. The proof of these statements is quite general. Experimental inferences are drawn.

M. J. Moravcsik (Livermore, Calif.)

7005:

Sitenko, A. G. Fission of nonspherical nuclei. Ž. Èksper. Teoret. Fiz. **36** (1959), 793–797 (Russian); translated as Soviet Physics. JETP **9**, 558–561.

This paper discusses a mechanism of direct fission caused by absorbing enough angular momentum into a deformed nucleus so that the centrifugal plus electrostatic forces overcome the surface tension causing immediate fission. A calculation of the cross section for this process for incident nucleons and deuterons is performed.

N. S. Wall (Cambridge, Mass.)

7006:

Power, E. A.; and Zienau, S. Coulomb gauge in non-relativistic quantum electrodynamics and the shape of spectral lines. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 427–454.

The authors discuss some inconsistencies and ambiguities in the usual treatment of non-relativistic radiative transitions calculated in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. It is shown that, unless careful examination of the matrix

elements is made, unwanted instantaneous static contributions appear in expressions essentially dealing with retarded interactions. The scattering of two electrons, the electric dipole transition matrix, the natural shape of spectral lines and resonant absorption and stimulated emission rates are given as examples which show this type of inconsistent behaviour.

Either by a separation of the part of the matrix element which is singular at $k=0$, or by use of a transformed interaction Hamiltonian, this difficulty can be avoided. The interaction Hamiltonian preferred is, in the electric dipole approximation, $-\sum_l e_l \mathbf{q}^{(l)} \cdot \mathbf{E}^{\perp}(0)$, a result long known. This Hamiltonian is obtained by a canonical transformation from the usual expression and a multipole expansion incorporating higher electric and magnetic moment terms is found. The transformation depends on the explicit use of the polarization field operator $\mathbf{P}$ which incorporates all the static terms, leaving the retarded radiative terms in the new electric and magnetic field operators.

The paper contains critical comments on previous work on these problems. *C. A. Hurst* (Adelaide)

7007:

Skillman, Sherwood. Efficient method for solving atomic Schroedinger's equation. Math. Tables Aids Comput. **13** (1959), 299–302.

A description, with specimen results, of a machine program for solving the Schroedinger equation for a spherical potential, using standard methods.

D. F. Mayers (Oxford)

7008:

el Meligy, Abd el Sadek; and el Sherbini, Mahmoud Ahmed. Energy levels for the Coulomb potential with cut-off. Z. Angew. Math. Phys. **10** (1959), 474–477. (German summary)

7009:

Pekeris, C. L. $1\,{}^1S$ and $2\,{}^3S$ states of helium. Phys. Rev. (2) **115** (1959), 1216–1221.

The method introduced by Pekeris [same Rev. **112** (1958), 1649–1658; MR **21** #591] is applied to the ground state of helium using a determinant of order 1078 and to the lowest triplet metastable state using a determinant of order 715. The non-relativistic eigenvalues are given to accuracies of 1 part in 10^{10} for the ground state and 1 part in 10^{12} for the metastable state. Including mass-polarization, relativistic and radiative corrections, the theoretical ionization potential of the ground state is 198310.687 cm^{-1} compared to the experimental value 198310.82 ± 0.15 cm^{-1} and the theoretical ionization potential of the triplet metastable state is 38454.66 cm^{-1} compared to the experimental value 38454.73 ± 0.05 cm^{-1}. For the metastable state, the theoretical electron density at the nucleus is 33.18416 compared to the experimental value 33.18388 ± 0.00023. *A. Dalgarno* (Belfast)

7010:

Birbrair, B. L.; Peker, L. K.; and Sliv, L. A. Quadrupole oscillations of deformed nuclei. Ž. Èksper. Teoret. Fiz. **36** (1959), 803–809 (Russian); translated as Soviet Physics. JETP **9**, 566–570.

The problem of finding the eigenvalues of the Bohr Hamiltonian for an axially symmetrical nucleus with n nucleons outside a core is solved for both the β and γ vibrations. The corrections to the rotational spectra because of the existence of these oscillations are also calculated. Comparison is made of experiment with the predictions of these calculations.

N. S. Wall (Cambridge, Mass.)

7011:

Bolsterli, M. Finite-range delta-function potential. Phys. Rev. (2) **114** (1959), 1605–1608.

A nuclear potential which is not strongly attractive at small distances and therefore is suitable for perturbation calculations is suggested for the nucleon-nucleon problem. The various parameters of the neutron-proton system are calculated with their potential and compared with experiment. The most serious question raised in their comparison is the relatively large ($\sim 8\%$) D state admixture. Of course the usual 4% D state admixture may result from the over-simplified nuclear model used to calculate the quadrupole and magnetic dipole moment. The solutions of the Schroedinger equation for the proposed δ function potentials are given in a useful appendix.

N. S. Wall (Cambridge, Mass.)

7012:

Jancovici, Bernard. Contribution à l'étude du role des forces non centrales dans la structure nucléaire. Ann. Physique **4** (1959), 689–744.

Le propos de l'article est l'étude des divers aspects des forces non centrales dans la structure des noyaux atomiques et les relations possibles entre ces forces.

Dans la première partie, l'auteur tente d'expliquer la longue durée de vie β de ^{14}C. Celle-ci est permise par la forme du spectre et les spins. La présence de forces centrales et de forces spin-orbite

$$a \sum_i \mathbf{l}_i \cdot \mathbf{S}_i \quad \text{et} \quad V(r_{12}) \mathbf{l}_{12} \cdot (\mathbf{S}_1 + \mathbf{S}_2)$$

ne peut annuler l'élément de matrice de la transition. Par contre, l'introduction de forces tensorielles convenablement ajustées annule cet élément de matrice. Il faut cependant que cette force ait une valeur assez faible pour ne pas perturber le spectre des états de ^{14}N.

Le même modèle explique la durée de vie β de ^{14}O dans sa transition vers l'état fondamental de ^{14}N et la durée de vie γ du premier état excité de ^{14}N.

Dans la deuxième partie, l'auteur montre que l'on peut expliquer les durées de vie β de ^{12}B et ^{8}Li par la présence de forces vectorielles (spin-orbite). Ces transitions β ont, en effet, lieu grâce au mélange partiel, par des forces non centrales, entre les supermultiplets. (Les fonctions d'ondes sont mieux décrites par le modèle des couches à couplage LS que par le couplage $j-j$). A l'intérieur de la configuration p^4 les forces tensorielles ne peuvent pas expliquer le mélange des supermultiplets. Il est cependant possible que les forces vectorielles possibles soient un effet du deuxième ordre des forces tensorielles.

Dans la troisième partie, l'auteur montre que pour expliquer les forces spin-orbite dans les noyaux complexes il faut admettre, sous réserve de la validité de certaines approximations, l'existence de forces élémentaires spin-orbite entre deux nucléons. Il demeure cependant la possibilité que les forces spin-orbite soient dues à des effets

des forces tensorielles mettant en jeu plus de deux nucléons à la fois. Ces conclusions découlent du fait qu'il est très improbable que les forces spin-orbite dans les noyaux puissent être expliquées comme un effet de deuxième ordre des forces tensorielles.

P. Chevallier (Strasbourg)

7013:

Blair, John S. Inelastic diffraction scattering. Phys. Rev. (2) **115** (1959), 928–938.

7014:

Fujii, A.; and Primakoff, H. Muon capture in certain light nuclei. Nuovo Cimento (10) **12** (1959), 327–355. (Italian summary)

7015:

Kumar, K. Statistical derivation of the nuclear rotational energies. Nuovo Cimento (10) **13** (1959), 591–604. (Italian summary)

7016:

Seaton, M. J. Radiative recombination of hydrogenic ions. Monthly Not. Roy. Astr. Soc. **119** (1959), 81–89.

"Using the first three terms in the asymptotic expansion of the Kramers-Gaunt factor, calculations are made for the rate of recombination and for the mean kinetic energy of the recombining electrons." *Author's summary*

7017:

Berencz, F. Einige Zweizentrenintegrale zu Rechnungen auf Grund der Methode der korrelationsmässigen Molekülbahn. Acta Phys. Acad. Sci. Hungar. **9** (1958/59), 381–392.

Numerical values are given for the following integrals:

$$I^n_{iklm} = \int \psi_1{}^2(\psi_2+\psi_3+\psi_4+\psi_5)^2 \mu_1{}^i \mu_2{}^k \nu_1{}^l \nu_2{}^m r_{12}{}^n d\tau,$$

$$J^n_{iklm} = \int \psi_1{}^2(\psi_2+\psi_3+\psi_4+\psi_5)(\psi_2 \pm \psi_3 - \psi_4 \mp \psi_5) \times \mu_1{}^i \mu_2{}^k \nu_1{}^l \nu_2{}^m r_{12}{}^n d\tau,$$

where $\psi_1 = \exp[-\alpha(\mu_1+\mu_2)]$, $\psi_2 = \exp[\beta(\nu_1+\nu_2)]$, $\psi_3 = \exp[\beta(\nu_1-\nu_2)]$, $\psi_4 = \exp[-\beta(\nu_1+\nu_2)]$, $\psi_5 = \exp[-\beta(\nu_1-\nu_2)]$; $i, k, l, m, n = 0, 1, 2$; $\alpha, \beta = .5(.25)2.5$; and r_{12} is the distance of the two electrons.

The method and basic integrals used in the calculation are given by M. Kotani, A. Amemiya and T. Simose [Proc. Phys.-Math. Soc. Japan **20** (1958), suppl.; **22** (1940), suppl. 1]. *I. A. Stegun* (Washington, D.C.)

7018:

Keffer, F.; and Oguchi, T. Theory of superexchange. Phys. Rev. (2) **115** (1959), 1428–1434.

The Dirac-Van Vleck-Serber spin-operator expansion is applied to the super-exchange theory of anti-ferromagnetism. The linear cation-anion-cation four-electron problem is worked out in detail using the configurations (A) ionic, (B) electron transferred to the right, (C) electron transferred to the left. This leads to several important new terms in addition to those previously discussed. All super-exchange terms are of at least the fourth order in the overlap integral. A numerical evaluation with Slater functions appropriate to Mn-O type crystals yields a super-exchange integral in approximate agreement with experiment. The linear cation-anion-anion-cation system is also considered and it is shown that for the ionic configuration super-exchange terms first appear in the order S^4T^2 where S is the anion-cation overlap and T is the anion-anion overlap. *A. C. Hurley* (Melbourne)

7019:

Samuel, Isaac. Application de la méthode graphique des polygones aux opérations sur les fonctions d'onde de Slater à orbitales réelles non orthonormées (méthode de glissement). Cahiers de Phys. **13** (1959), 191–195.

Each term in the expansion of a determinant may be characterized by an order of the indices. It is shown how tables prepared in this way may be used to facilitate the calculation of matrix elements between Slater determinants constructed from non-orthogonal orbitals.

A. C. Hurley (Melbourne)

7020:

Mizuno, Yukio; and Izuyama, Takeo. Consideration on non-orthogonality catastrophe in the Heitler-London theory. I. Progr. Theoret. Phys. **22** (1959), 344–350.

It is shown that, if the overlap integrals in the Heitler-London theory of many-electron systems satisfy certain inequalities, then the non-orthogonality catastrophe suggested by D. R. Inglis [Phys. Rev. (2) **46** (1934), 135–138] will not occur. This result is also established for the electron pair bond (or paired-electron orbital) theory in which overlap integrals between bonded orbitals are retained while other overlap integrals are neglected.

A. C. Hurley (Melbourne)

7021:

Viswanathan, K. S. The relativistic theory of chemical binding. Proc. Indian Acad. Sci. Sect. A **50** (1959), 1–18.

The relativistic theory of the electronic states of the hydrogen molecule is developed from the Breit equation. It is shown that the component of the total angular momentum parallel to the nuclear axis is a constant of the motion. The usual non-relativistic formulation is obtained by neglecting the spin-orbit coupling and other small terms in the Schrödinger-Pauli form of the Breit equation.

A. C. Hurley (Melbourne)

7022:

Mazur, J.; and Rubin, Robert J. Quantum-mechanical calculation of the probability of an exchange reaction for constrained linear encounters. J. Chem. Phys. **31** (1959), 1395–1412.

"A numerical procedure suitable for use with a high-speed computing machine is developed for calculating the average quantum-mechanical probability of the exchange reaction $BC+A \rightarrow B+CA$ for constrained linear encounters at temperature T when BC is initially in its ground or first excited vibrational state. The average refers to the average over the relative momentum frequency distribution of collisions between BC and A at temperature T. The procedure, which involves the numerical solution of the time-dependent Schrödinger equation, is

sufficiently general so that any three-atom potential energy surface may be used. Two sample calculations have been performed using a simple potential energy surface. The results of these quantum-mechanical calculations for the average probability of reaction are compared with the corresponding classical quantities which are obtained in an elementary fashion." (Author's abstract)

A. C. Hurley (Melbourne)

7023:

Jones, Walter D. Hückel theory: an effective Hamiltonian. J. Chem. Phys. **31** (1959), 1317–1319.

"An effective one-electron Hamiltonian for molecules is presented that gives exactly the single-atom energies for its diagonal elements. An operator is used that selectively cancels the nuclear attraction for all nuclei except the one on which a particular *AO* is centered." (Author's abstract) *A. C. Hurley* (Melbourne)

7024:

Widom, B. Mean-first-passage times and the collision theory of bimolecular reactions. J. Chem. Phys. **31** (1959), 1387–1394.

A general collision theory is outlined for the kinetics of reaction of molecules which are dilutely dispersed in an inert gas, reaction being the result of binary collisions between reacting molecules and inert gas molecules. The mean-first-passage time is obtained from the solution of an integral equation and it is shown that if a rate constant exists it is the reciprocal of the mean-first-passage time. The equilibrium hypothesis is analysed and the theory is illustrated with two simple transition kernels.

A. C. Hurley (Melbourne)

7025:

Sakurai, J. J. Useful symmetries of strong interactions. Phys. Rev. (2) **115** (1959), 1304–1309.

An attempt is made to obtain "useful" relations among strange particle reactions, stronger than those implied by charge independence but not so strong as the Pais S_1-S_2 rule. In particular, if it is assumed that

$$G_{K\Lambda N} = G_{K\Sigma N}, \quad G_{K\Xi\Lambda} = G_{K\Xi\Sigma}$$

and that the relative parity of Λ and Σ is even, and further that $G_{\Pi\Lambda\Sigma} = \mp G_{\Pi\Sigma\Sigma}$, then it is sufficient for obtaining such relations to impose the dynamical assumption that pions emitted by N and Ξ [Λ and Σ] be not absorbed by Λ and Σ [N and Ξ]. *P. Roman* (Manchester)

7026:

Frazer, William R. Proposal for determining the electromagnetic form factor of the pion. Phys. Rev. (2) **115** (1959), 1763–1772.

The paper gives a method of determining the electromagnetic form factor of the pion from an extrapolation of experiments on electron production of pions from nucleons. The form factor is proportional to the residue of the pole in the pion-electroproduction amplitude viewed as a function of the invariant momentum transfer of the nucleon. Since the pole is outside the physical region for pion-electroproduction, an extrapolation is necessary which, it is estimated, cannot be carried out significantly without an accuracy in the data that is an order of magnitude better than that available at the beginning of 1959. *M. J. Moravcsik* (Livermore, Calif.)

7027:

Arzhanykh, I. S. The differential equations of motion of a meson charge. Dokl. Akad. Nauk SSSR **125** (1959), 1215–1218 (Russian); translated as Soviet Physics. Dokl. **4**, 298–302.

7028:

Emel'yanov, A. A. Viscosity in the hydrodynamic theory of multiple particle production. Ž. Èksper. Teoret. Fiz. **36** (1959), 1550–1554 (Russian); translated as Soviet Physics. JETP **9**, 1100–1102.

The model of a viscous ultrarelativistic fluid is used to describe the dispersion of the meson-nucleon cloud produced in the collision of high-energy nucleons. An asymptotic solution of one-dimensional equations is obtained. It is shown that when viscosity is taken into account the angular distribution of secondary particles is less anisotropic than in the case of an ideal fluid. *Author's summary*

7029:

Takabayasi, T. Relativistic particle with internal rotational structure. Nuovo Cimento (10) **13** (1959), 532–554. (Italian summary)

The general kinematical and dynamical aspects of any covariant theory of elementary objects possessing internal angular velocity and angular momentum are analyzed. The kinematical representative of such a structure is a spinor and a precise theory is constructed by postulating a nonlinear equation of motion which contains two structure constants of the "particle". The equation can be solved exactly and it is shown that the particle performs, besides its mean rectilinear motion, an orbital "Zitterbewegung" around the direction of the constant momentum vector and a synchronized precession of the internal third axis around the constant helicity pseudovector.

P. Roman (Manchester)

7030:

Feinberg, Gerald; and Gürsey, Feza. Space-time properties and internal symmetries of strong interactions. Phys. Rev. (2) **114** (1959), 1153–1170.

G. Feinberg showed in a previous paper that P invariance of the π-nucleon interaction follows from CP invariance if the interaction is of Yukawa type (with non-derivative coupling) and possesses charge symmetry [G. Feinberg, same Rev. **108** (1959), 878–881; MR **19**, 1132]. In the present paper generalizations of this result to include all baryons and K-mesons are considered. The basic idea is the same: CP invariance plus special "internal symmetries" of the interactions or "hidden gauge principles" ought to be responsible for P invariance. The generalizations require that a doublet structure of all baryons exist in their Yukawa interactions with π and K mesons and that the interactions possess certain symmetries corresponding to reflections in an internal space. The resulting set of interactions leads to selection rules in conflict with experiment and the departures from production are attributed to quadrilinear couplings for which

P conservation is again a consequence of CP and other symmetries. The results are too complicated to summarize here. *A. S. Wightman* (Princeton, N.J.)

7031:

Matthews, P. T.; and Salam, A. The inelastic scattering of elementary particles. Nuovo Cimento (10) **13** (1959), 381–393. (Italian summary)

This paper discusses the restrictions imposed by unitarity and causality on the low energy inelastic scattering of elementary particles in the case where many channels, each containing just two particles, are open. It is shown that the scattering amplitudes may be expressed in terms of two constant matrices which are the generalizations to many channels of the scattering length and effective range which occur in single channel theory. The Breit-Wigner and Chew-Low formulas are derived from the same point of view. By a further application of unitarity the parameters which describe scattering below threshold for one or more of the channels are related to the larger number needed when all channels are open.

P. W. Higgs (London)

7032:

Pac, Pong Y. On the transformation properties of the Dirac equation. Progr. Theoret. Phys. **21** (1959), 640–652.

For a spin 1/2 particle the helicity operator $h=\sigma\cdot\mathbf{p}/(p)$ has unit square and anticommutes with the space inversion operator P. Consequently the operators P, h and ihP form a group isomorphic to the Pauli group. For the Dirac equation one can define another group in which h is replaced by εh where ε is the sign of the energy. The author introduces these two groups and calls them the helicity group and the pseudohelicity group, respectively. A class of unitary transformations of the type first studied by Foldy and Wouthuysen [Phys. Rev. (2) **78** (1950), 29–36] and by Tani [Progr. Theoret. Phys. **6** (1951), 267–285; MR **13**, 414] is discussed. The relevance of these more general transformations is certainly not apparent from this paper. *E. C. G. Sudarshan* (Rochester, N.Y.)

7033:

Granovskii, Ya. I. On the properties of K-mesons. Dokl. Akad. Nauk SSSR **125** (1959), 1225–1226 (Russian); translated as Soviet Physics. Dokl. **4**, 400–401.

At the present time, the properties of K-mesons are studied experimentally. In this note we show that such properties of K-mesons as mass, spin, and parity, may be immediately obtained from Heisenberg's theory [Rev. Mod. Phys. **29** (1957), 269–278; MR **19**, 813].

Author's summary

7034:

May, R. M. Superconductivity of a charged ideal 2-dimensional Bose gas. Phys. Rev. (2) **115** (1959), 254–262.

A perfect two-dimensional Bose gas does not have a phase transition like a three-dimensional Bose gas has. A charged three-dimensional Bose gas in its condensed state completely expels a weak magnetic field, so the author investigates how the two-dimensional charged gas behaves in a magnetic field. It is found that, below a certain critical temperature, the two-dimensional gas expels almost all the magnetic field, and the expulsion is complete at absolute zero. The penetration of the field is calculated, and also the critical field needed to destroy this effect is found. *D. J. Thouless* (Birmingham)

7035:

Cooper, L. N.; Mills, R. L.; and Sessler, A. M. Possible superfluidity of a system of strongly interacting fermions. Phys. Rev. (2) **114** (1959), 1377–1382.

The condition for superfluidity of a system of interacting fermions is investigated. If the particles can be correlated in pairs, as they are in the Bardeen, Cooper, and Schrieffer theory of superconductivity, to give a state with a lower energy than the uncorrelated state has, then the system is expected to be superfluid. The condition for this correlation to be possible is expressed in a variational form. Several functions are tried out to see if the variational principle can be satisfied for nuclear matter or for liquid He^3, but no function is found which comes close to satisfying it. {Later work by these authors and by others has shown that the variational principle can be satisfied for both systems.} *D. J. Thouless* (Birmingham)

7036:

Ch'ên, Ch'un-hsien. A method for taking into account correlation in a many-particle system. Dokl. Akad. Nauk SSSR **125** (1959), 1238–1241 (Russian); translated as Soviet Physics. Dokl. **4**, 413–416.

The n-particle Green's functions for a many-fermion system can be expressed in terms of correlation functions of n particles and less. The coupled equations for the Green's functions then lead to coupled nonlinear equations for the correlation functions. If correlations of two particles or more are ignored, the theory of the self-consistent field is obtained. If correlations of more than two particles are neglected, Brueckner's theory of nuclear matter, the theory of correlation in an electron gas, the dispersion law for zero sound, and the modern theory of superconductivity can be derived.

D. J. Thouless (Birmingham)

7037:

Klein, Abraham. Many-particle approach to the one-electron problem in insulators and semiconductors. Phys. Rev. (2) **115** (1959), 1136–1146.

The formal techniques of second quantization (many-fermion theory) are applied to the discussion of the motion of electrons in solids, taking into account the polarizability of the matter.

E. L. Hill (Minneapolis, Minn.)

7038:

Eyges, Leonard. Quantum-mechanical three-body problem. Phys. Rev. (2) **115** (1959), 1643–1655.

A system of three identical spinless particles is considered, neglecting symmetry conditions. Instead of using a potential energy, boundary conditions for the logarithmic derivative of the wave function are introduced. It is shown that these boundary conditions are roughly equivalent to an additive potential energy with hard core and

strong short range attractive two-body forces. The resulting equations can be treated by partial wave expansions and can often be truncated. The case of the ground state is discussed, taking S-waves only into account.

D. ter Haar (Oxford)

7039:

Lee, T. D.; and Yang, C. N. Many-body problem in quantum statistical mechanics. II. Virial expansion for hard-sphere gas. Phys. Rev. (2) **116** (1959), 25–31.

The method of part I [same Rev. **113** (1959), 1165–1177; MR **21** #1725] is used to evaluate the coefficients in the expansion in terms of the fugacity of the equation of state of a system of hard-sphere bosons of fermions.

D. ter Haar (Oxford)

7040:

Shono, Yoshiyuki; and Tanaka, Hajime. Nuclear collective motion and the effective two-body potential. Progr. Theoret. Phys. **22** (1959), 177–191.

It is investigated in how far an independent particle model plus an effective two-body potential may account for the nuclear deformation and rotational motion of nuclei. The effective potential is determined in a self-consistent way and consists of two parts: Elliott's Q-Q interaction and a restoring term. A two-dimensional oscillator model is used. The rotational levels are calculated and the moment of inertia turns out to be about one third of its rigid value. *D. ter Haar* (Oxford)

7041:

Nakajima, Sadao. A note on the electromagnetic response of superconductors. Progr. Theoret. Phys. **22** (1959), 430–436.

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See also 6984, 7021, 7029, 7076.

7042:

★Fock, V. The theory of space time and gravitation. Translated by N. Kemmer. Pergamon Press, New York-London-Paris-Los Angeles, 1959. xviii+411 pp. $15.00.

This is a translation of *Teoriya prostranstva, vremeni i tyagoteniya* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955], reviewed in MR **18**, 445. The author collaborated in the present translation and effected a few changes for it, such as altering some mathematical proofs and making some reasonings more precise; he also contributed a preface.

7043:

Odagiri, Mizuho. On some fundamental problems of the theory of relativity. J. Osaka Inst. Sci. Tech. Part I **8** (1958), no. 2, 1–12.

7044:

Costa de Beauregard, Olivier. L'hypothèse de l'effet inertial de spin. Cahiers de Phys. **13** (1959), 200–208.

The author proposes an experiment to test whether magnetising a piece of iron gives it an acceleration, as would be expected in his theory of spinning matter [C. R. Acad. Sci. Paris **247** (1958), 1092–1094; MR **20** #7577]. *D. W. Sciama* (London)

7045:

Dobresku-Puriče, Lučiya. Geometrization of the phenomenon of propagation by waves, defined by a partial differential equation of second order in space time. Rev. Math. Pures Appl. **3** (1958), 413–426. (Russian)

A Romanian version appears in Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 401–414 [MR **21** #3192].

7046:

Metz, André. Les phénomènes "stationnaires" dans les mouvements en circuit fermé. C. R. Acad. Sci. Paris **249** (1959), 1460.

7047:

Jones, Robert T. Extending the Lorentz transformation by characteristic coordinates. Amer. J. Phys. **28** (1960), 109–111.

"The problem considered is that of rectilinear motion with variable velocity. The paper gives, by an elementary construction, a system of coordinates which is conformal in a restricted region near the axis of the motion. In such coordinates the velocity of light remains invariant even for observers moving with variable velocity. By a particular choice of the scale relation the restricted conformal transformations can be made to reduce to the Lorentz transformation everywhere in the case of constant velocity and locally in the case of variable velocity."

Author's summary

7048:

Stiegler, K. On the mechanical foundation of the theory of special relativity. Nuovo Cimento (10) **13** (1959), 873–879. (Italian summary)

7049:

Builder, G. The Lorentz transformations. Austral. J. Phys. **12** (1959), 300–303.

This paper is a rejoinder to criticism by Sir Harold Jeffreys [same J. **11** (1958), 583–586; MR **21** #1878] that the author's analysis of the clock paradox [ibid. **10** (1957), 246–262; MR **19**, 813] is based on concealed hypotheses.

G. J. Whitrow (London)

7050:

Zăgănescu, M. Hodographe du movement du point matériel dans la mécanique rélativiste. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 195–200 (1959). (Romanian. French and Russian summaries)

On considère le mouvement bidimensionel et on deduit l'équation de l'hodographe des équations du mouvement du point matèriel. Cette équation est du type Riccatti. Si les composantes de la force sont constantes, le hodographe représente une equidistante du plan de Lobatschewski.

Résumé de l'auteur

7051:

Builder, Geoffrey. The fundamental concepts of relativity theory. Austral. J. Sci. **22** (1959), 87–97.

The author discusses conceptions and misconceptions concerning the special theory of relativity. It is not feasible to summarize his discursive arguments in a review of reasonable length. *H. A. Buchdahl* (Princeton, N.J.)

7052:

Rzewuski, J. The spinor space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 261–269. (Russian summary, unbound insert)

The author proposes to regard physical space as an eight-dimensional spinor space S_8 rather than a four-dimensional vector space E_4 on the grounds that the spinor space is contained in a natural way in the mathematical framework of the Lorentz group; and that it is richer than the vector space in transformations not violating the principle of relativity. The hermitian spinor $z_{\alpha\dot{\beta}}$ associated with the real vector X_μ is taken to be a bilinear form in the four complex variables z_α, $z_{\dot{\beta}}$, the eight-dimensional space of these variables being assumed to be physical space. The following topics are investigated: (i) the relations between invariants in E_4 and S_8; (ii) space and time inversions; (iii) movements of S_8 which leave E_4 invariant. Charge and isospin are briefly referred to.

H. A. Buchdahl (Princeton, N.J.)

7053:

Capps, Richard H. Behavior of intense relativistic electron beams. Phys. Rev. (2) **114** (1959), 1203–1217.

A beam of fast electrons, which can accumulate positive ions, is under certain conditions magnetically self-focussing [W. H. Bennett, same Rev. **98** (1955), 1584–1593], and for certain values of the parameters describing the beam there exists an "equilibrium state", in which the radius and mean electron energy of the beam are constant in time [G. J. Budker, Proc. CERN Sympos. High Energy Accelerators Pion Phys., Geneva, 1956, pp. 68–75, CERN, Geneva, 1956]. The theoretical investigation in this paper considers in detail the processes by which a non-equilibrium relativistic beam of electrons could eventually be brought into the equilibrium state in a reasonably short interval of time. The stability of the beam is not investigated. Equations are derived which give the variation with time of the radius and mean electron energy of the beam in terms of instantaneous parameters of the beam, the densities of the electrons, ions and neutral atoms, and the impressed electric and magnetic fields. The time for shrinkage of the radius of the beam to the equilibrium value is influenced by variation of the impressed fields, electron radiation damping and scattering processes. The equations are integrated for certain sets of values of the parameters to determine the radius and mean energy as functions of the time. In these cases it is found that the time necessary for the radiative effect to shrink the beam to the equilibrium value is of the order 10^{-2} to 10^{-1} sec.

C. Gilbert (Newcastle-upon-Tyne)

7054:

Papapetrou, A.; und Treder, H. Zur Frage der Existenz von singularitätsfreien Lösungen der allgemein-relativistischen Feldgleichungen, die Teilchenmodelle darstellen könnten. Ann. Physik (7) **3** (1959), 360–372.

Einstein's old question concerning the existence of solutions free of singularities, vanishing and becoming Euclidean at infinity is generalized here for the case of periodic solutions free of singularities. If they existed they could represent particles. The result is negative; such solutions do not exist. Nor do they exist if we take the Einstein-Maxwell equations into account.

L. Infeld (Warsaw)

7055:

Zatzkis, Henry. Model of a linear harmonic oscillator in the general theory of relativity. Phys. Rev. (2) **114** (1959), 1645–1647.

The author investigates the radial oscillatory motion of a test particle moving in a space whose metric is that of the standard Schwarzschild interior solution. {The "frequency" of the motion appears to be a formal quantity relating to coordinate time.}

H. A. Buchdahl (Princeton, N.J.)

7056:

Brill, Dieter R. On the positive definite mass of the Bondi-Weber-Wheeler time-symmetric gravitational waves. Ann. Physics **7** (1959), 466–483.

Consider a solution $g_{\alpha\beta}$ of the field equations $R_{\alpha\beta}=0$ which has the property of "time-symmetry about $t=0$", viz.

$$g_{ij}(t, x_l) = g_{ij}(-t, x_l);$$

$$g_{00}(t, x_l) = g_{00}(-t, x_l); \quad g_{i0}(t, x_l) = -g_{i0}(-t, x_l);$$

where $i, j = 1, 2, 3$; $\alpha, \beta = 0, 1, 2, 3$. Then the author shows that in the coordinate system for which $g_{00} = -1$, $g_{0i} = 0$, the initial-value problem reduces to the solution of

$$(1) \qquad {}^3R = 0$$

where ${}^3R$ is the curvature scalar of the three-space g_{ij}.

The solutions of (1) for axially-symmetric, everywhere regular g_{ij} are next investigated. Assuming the g_{ij} to be asymptotically expansible in power series in r^{-1}, the author shows that the g_{ij} contain for large r a term of order m/r such that $m \geqq 0$, and the equality holds only if g_{ij} is flat. Procedure for the construction of such solutions, and a proof of their existence, are given.

It is concluded that fields of pure gravitational radiation contain energy and mass. *W. B. Bonnor* (London)

7057:

Costa de Beauregard, Olivier. L'hypothèse de l'effet gravitationnel de spin. Cahiers de Phys. **13** (1959), 209–216.

The author further studies his idea [C. R. Acad. Sci. Paris **214** (1942), 904–906; MR **5**, 131] that spinning matter produces a non-symmetric gravitational field, and compares his results with those of the reviewer [Proc. Cambridge Philos. Soc. **54** (1958), 72–80; MR **20** #727].

D. W. Sciama (London)

7058:

Pratelli, Aldo M. Sulla libertà secondo Einstein delle equazioni di campo. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 251–291.

Let a set $\{U\}$ of N unknown functions in a n-dimensional space be subject to e differential equations (of order r) which are related by i differential relations of order $s \geqq r$.

It is not difficult to find out the number $\Omega(n, m, N, e, i, r)$ of coefficients which may be chosen freely at the mth step of the Taylor expansion of U. Einstein derived this number under the name of "measure of freedom" for $n=4$ [*The meaning of relativity*, 4th ed., Princeton Univ. Press, 1953; MR **14**, 805; Appendix II]. The author extends this result for an arbitrary n. Rewriting skillfully the formula for Ω he obtains an asymptotic expression for large m as a sum of three numbers, one of them being Einstein's coefficient of freedom [loc. cit.] in the case $n=4$. Then he applies this formula to more than fifteen different examples from the realm of classical and relativistic physics. {Reviewer's remark: The following counterexample shows that the above mentioned method is not a panacea for all cases. If $\mathscr{L}_{a\lambda^r}$ $(a=1, \cdots, r)$ are the generating tensors of the holonomy group G_r of a space L_n with symmetric connection, then

$$\mathscr{L}_{a\lambda^r} w_r = 0 \tag{1}$$

is a complete set of integrability conditions of

$$\nabla_\mu w_\lambda = 0. \tag{2}$$

If the rank of (1) is $p<n$, then (2) admits $q=n-p$ linearly independent solutions w_λ defined by the initial conditions $w_\lambda(P)$ at P. [See Hlavatý, J. Math. Mech. **8** (1959), 597–622; MR **21** #6002.] Hence here we have $\Omega=ng$ for any m. However, this result can not be derived without the knowledge of the $\mathscr{L}_{a\lambda^r}$.} *V. Hlavatý* (Bloomington, Ind.)

7059:

Witten, Louis. Geometry of gravitation and electromagnetism. Phys. Rev. (2) **115** (1959), 206–214.

The author uses the spinor calculus to re-derive the already unified theory of Rainich [Trans. Amer. Math. Soc. **27** (1925), 106–136] and Misner and Wheeler [Ann. Physics **2** (1957), 525–603; MR **19**, 1237]. He also obtains algebraic relations between the Ricci tensor, the electromagnetic field tensor and their principal null vectors. A null electromagnetic field is an exceptional case, whose status in already unified theory is not clear.

D. W. Sciama (London)

7060:

Debever, Robert. Sur le tenseur de super-énergie. C. R. Acad. Sci. Paris **249** (1959), 1324–1326.

The Bel-Robinson tensor [see L. Bel, C. R. Acad. Sci. Paris **248** (1959), 1297–1300; MR **21** #1194] is expressed in terms of four null vectors associated with the Riemann tensor in a hyperbolic normal Riemann space V_4 with zero Ricci tensor (i.e., a vacuum space-time). The arrangement of the null vectors determines the Petrov type [A. Z. Petrov, Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 55–69; MR **17**, 892]. The occurrence of k coincident null vectors is denoted by a k in the symbol [ ] in the following table:

Type:	I	D	II	III	N
Symbol:	[1111]	[211]	[22]	[31]	[4]

(Here D denotes degenerate type I, with two equal eigenvalues, and N denotes null type II, with zero eigenvalues.)

{A similar characterization was reported by R. Penrose at the Royaumont conference on gravitation, June, 1959. [See R. Penrose, Ann. Physics **10** (1960), 171–201.]}

F. A. E. Pirani (London)

7061:

Peres, Asher; and Rosen, Nathan. Nonlinear effects of gravitational radiation. Phys. Rev. (2) **115** (1959), 1085–1086.

It is shown that the nonlinear terms in Einstein's equations rule out the possibility of stable small oscillations of a gravitational field about an equilibrium state (supposed Minkowskian at infinity). The perturbation tends to take infinite values at large distances from its source. This can be interpreted as an instability of gravitational radiation fields, and raises doubts concerning the validity of the linearized theory; though an estimate of the nonlinear effects shows that, for any reasonable values of the material sources, they become important only at distances which are much larger than the dimensions of the universe.

C. W. Kilmister (London)

7062:

Buchdahl, H. A. Reciprocal static metrics and scalar fields in the general theory of relativity. Phys. Rev. (2) **115** (1959), 1325–1328.

It is shown that to every static solution of the Einstein gravitational field equations for empty space there corresponds a one-parameter family of solutions of the field equations for the case in which a certain scalar field is present. This can be regarded as an extension of the author's earlier work on "reciprocal solutions" [see, e.g., Austral. J. Phys. **9** (1956), 13–18; MR **18**, 704] and is related to the papers on scalar fields by O. Bergmann and R. Leipnik [Phys. Rev. (2) **107** (1957), 1157–1161; MR **19**, 1021] and by H. Yilmaz [ibid. **111** (1958), 1417–1426; MR **21** #5066]. The motion of a test particle in the scalar field is investigated. The cases of the spherically symmetric and axially symmetric fields are treated explicitly, and the physical significance of the former is discussed.

N. Rosen (Haifa)

7063:

Synge, J. L. A theory of elasticity in general relativity. Math. Z. **72** (1959/60), 82–87.

The classical Hooke's Law which reads: "Stress is a linear function of strain" is modified to become "rate of change of stress is a linear function of rate of strain". This reformulation of Hooke's Law is then expressed in mathematical symbols and, with the aid of Einstein's field equations, the mathematical theory of a gravitating elastic body is formulated, particularly in relation to the propagation of elastic waves. *L. Infeld* (Warsaw)

7064:

Boardman, John; and Bergmann, Peter G. Spherical gravitational waves. Phys. Rev. (2) **115** (1959), 1318–1324.

The field equations of the general relativity theory are solved in the linear approximation for spherical gravitational waves emitted by a quadrupole source. The method is based on the use of a "superpotential", somewhat analogous to the Hertz vector in electromagnetic theory. It is found that there exist four distinct classes of such waves, of which one is of the type obtained by N. Rosen and H. Shamir [Rev. Mod. Phys. **29** (1957), 429–431; MR **19**, 927]. By the use of the canonical expression for the energy flux it is found that energy is radiated outward

from the source in each case. As a check on the method, the authors carry out calculations on cylindrical waves to the same approximation and obtain results which correspond to those obtained by exact calculations.

N. Rosen (Haifa)

7065:

Kalitzin, N. St. Verallgemeinerung der Grundformel der relativistischen Mechanik für einige praktisch wichtige Nichtinertialsysteme. Nuovo Cimento (10) **13** (1959), 173–185. (Italian summary)

Two non-relativistic systems are two systems moving with small velocity relative to each other so that one can have a universal time and one can introduce rigid bodies. The special relativistic form of Newton's second law is investigated for such systems and it is shown that, as well as the centrifugal and Coriolis forces, a new fictitious force arises for the case when the particles move with high velocity. These results are compared with the more exact formulation of general relativity.

C. W. Kilmister (London)

7066:

Metz, André. Au sujet de la métrique relativiste sur un disque en rotation. Cahiers de Phys. **13** (1959), 196–200.

This is a popular essay on the famous problem of the geometry of a rotating disk in general relativity theory.

L. Infeld (Warsaw)

7067:

Peres, A. Gravitational motion and radiation. III. Nuovo Cimento (10) **13** (1959), 439–441. (Italian summary)

The change of the total gravitational mass of a system of two bodies, uniformly rotating around each other, is minus the radiated energy, calculated in a previous paper [Nuovo Cimento (10) **11** (1959), 644–655; MR **21** #3253].

C. W. Kilmister (London)

7068:

Kustaanheimo, Paul. Scalar field theory as a theory of gravitation. Soc. Sci. Fenn. Comment. Phys.-Math. **21** (1958), no. 3, 10 pp.

O. Bergmann [Amer. J. Phys. **24** (1956), 38–42] recently resuscitated the idea of a Lorentz invariant scalar potential theory of gravitation but was unable to obtain results comparable with those of general relativity for the three Einstein tests. In the present paper the author formulates a more general scalar potential theory in Minkowskian space-time. He obtains arbitrary numerical values for the results of the Einstein tests, but if he imposes the following axioms: (i) the velocity of light is a limit velocity which is never attained by a material particle; (ii) the proper mass of a particle is constant along every world line; (iii) the 'gravitational force' vector is a homogeneous polynomial of the first degree in the gradient of the potential; then there is the same gravitational red shift as in general relativity, but the perihelion advance is still arbitrary and there is no gravitational deflection of light.

G. J. Whitrow (London)

7069:

Kustaanheimo, Paul. On a unified field theory based on the special theory of relativity. Soc. Sci. Fenn. Comment. Phys.-Math. **21** (1958), no. 4, 11 pp.

A recent interpretation of Mach's principle that the inertia of a particle is due to a potential caused by other particles is developed by the author into a tensor potential theory in Minkowskian space-time representing the sum of all fields due to single particles, infinite self-potentials being omitted. The author claims that his theory accounts for inertia and for electromagnetic and gravitational forces. He obtains classical gravitational and electromagnetic theories in the first approximation and an arbitrary perihelion advance in the second approximation. He does not, however, obtain a universal constant of gravitation. Instead, he finds that it would be replaced by a parameter depending on the total potential of the universe in the region of the solar system and varying according to time and place. Similarly, the inertial mass of a particle would vary. The alleged effect would only be detectable by the Eötvös experiment if it could be performed with greater accuracy.

G. J. Whitrow (London)

7070:

Robinson, I. A solution of the Maxwell-Einstein equations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 351–352. (Russian summary, unbound insert)

L'auteur détermine une solution à symétrie sphérique des équations du champ électromagnétique-gravitationnel; elle correspond dans le cas des champs faibles à des champs électriques ou magnétiques constants.

J. Renaudie (Montpellier)

7071:

Datta Majumdar, S. Einstein's field equations. Nuovo Cimento (10) **13** (1959), 880–890. (Italian summary)

If the equations $G_{\mu\nu}=0$ are to have a consistent solution in the space surrounding an isolated body, the resultant force and torque on the body due to the field must vanish. A general solution for a static electro-gravitational field can be constructed by successive approximations.

L. Infeld (Warsaw)

ASTRONOMY

See also 7093.

7072:

★Eulerus, Leonhardus. Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. XXII. Theoria motuum lunae nova methodo pertractata. Edidit L. Courvoisier. Societas Scientiarum Naturalium Helveticae, Lausanne, 1958. xxx+412 pp.

In the 73 volumes of the edition now in course of reprinting Euler's previously published works, 10 are to collect his researches on celestial mechanics. The first of these is the volume under review. It contains "The theory of the motions of the moon", a book first published in 1772 as the culmination of Euler's struggle for forty years to master the intricacies of the lunar orbit. The calculations were carried out by J. A. Euler, Krafft, and Lexell. The editing was completed by the late L. Courvoisier in 1948. He checked in all detail the monstrous numerical calculation in which the work consists and found numerous errors, most of which, as often happens in Euler's

work, turned out to be of no importance in Euler's specific example for comparison of theory with observation. The editor has left Euler's values uncorrected except at the end results in each chapter, where not only correct numerical values but also a number of terms omitted in the original have been supplied. The lunar tables are reprinted in reduced facsimile.

In the editor's preface are clear, brief summaries of the ideas used in each chapter. There is also a brief but enlightening history of theories of the moon.

This volume furnishes a remarkable example of the scientific accuracy generally mantained by the Euler edition. *C. Truesdell* (Bloomington, Ind.)

7073:

Thüring, B. Programmgesteuerte Berechnung von Librationsbahnen. Astr. Nachr. **285**, 71–80 (1959).

Calculations with a high-speed computer are used to examine orbits about the equilateral libration centers L_4 and L_5 in the restricted problem of three bodies. The calculations are made with mass ratios ranging from $\mu = m_2/(m_1+m_2) = 0.0009539$, appropriate for the system Sun-Jupiter, to $\mu = 0.0123\ldots$, appropriate for the system Earth-Moon.

The emphasis is on orbits with large amplitudes of libration and on a spiral-type development of the libration orbits which is more evident for orbits with larger values of μ. No attempt is made to interpret these results analytically, but the conclusion is drawn that the absence of Trojan asteroids with very large libration amplitudes may be explained by the fact that orbits of this type have a short life time.

{The interpretation of the results is unfortunately complicated by the presence of short-period oscillations, with the period approximately equal to that of the period of revolution of the finite masses. Such oscillations are present in the general solution, but for a study of the long-period librations it would be desirable to make special efforts to deal with orbits which are essentially free from short-period effects.} *D. Brouwer* (New Haven, Conn.)

7074:

Jeffreys, Harold. Nutation: comparison of theory and observations. Monthly Not. Roy. Astr. Soc. **119** (1959), 75–80.

7075:

Ilinskiĭ, A. Yu. On the autonomous determination of position of a moving object by means of a space gyrocompass, a directional gyroscope and an integrating device. J. Appl. Math. Mech. **23** (1959), 75–82 (58–63 Prikl. Mat. Meh.).

L'auteur présente une étude théorique assez détaillée pour la détermination de la position des corps en mouvement aux moyens des gyroscopes et des accéléromètres.

M. Kiveliovitch (Paris)

7076:

Brumberg, V. A. Les corrections relativistiques dans la théorie de la lune. Byull. Inst. Teoret. Astr. **6**, 733–756 (1958). (Russian. French summary)

[This article was listed in MR **20** #6314.] C'est De Sitter qui a été le premier en 1916 pour étudier les corrections relativistes à la théorie du mouvement de la lune.

L'auteur étudie dans cet article, en détails, cette question.

L'auteur prend comme base le problème restreint circulaire des 3 corps de la théorie générale de la relativité, en 2ème approximation.

En appliquant les méthodes de Hill Brown, il retrouve les inégalités fondamentales de la relativité.

Dans le premier paragraphe, l'auteur déduit l'intégrale de Jacobi; dans le 2ème, la précession géodésique; dans le 3ème et le 4ème, une solution particulière périodique avec quelques corrections; dans le 5ème, il introduit les corrections qui dépendent de l'excentricité et de l'inclinaison. Enfin dans le dernier, il déduit les formules définitives pour les coordonnées de la lune ainsi que les valeurs numériques des corrections. *M. Kiveliovitch* (Paris)

7077:

Vinti, John P. New method of solution for unretarded satellite orbits. J. Res. Nat. Bur. Standards Sect. B **63B** (1959), 105–116.

A solution V of Laplace's equation is found which, on the one hand, makes the Hamilton-Jacobi equation separable and hence permits the reduction of the problem of motion in the field to quadratures, and on the other hand, by proper choice of two arbitrary constants, can be made to approximate, within observational error, the even zonal harmonics of the earth's gravitational field. The solution is

$$V = -\mu c^{-1}(\xi^2+\eta^2)^{-1}(\xi+\eta\delta/c),$$

where ξ, η are constant respectively over confocal oblate ellipsoids of revolution and confocal hyperboloids of revolution of one sheet; μ and c are constants, and δ is the displacement, along the axis of revolution, between the origin of coordinates and the center of mass.

The introduction of δ permits one, not only to take account of possible failures of the observers to place the origin of coordinates at the center of mass of the earth, but also to approximate the effects of other odd harmonics, for particular satellites.

The analysis is carried up to the statement of the kinetic equations of motion as quadratures.

J. A. O'Keefe (Chevy Chase, Md.)

7078:

Rüssmann, Helmut. Über die Differentialgleichungen eines dynamischen Weltmodells. Z. Astrophys. **48** (1959), 39–51.

The author obtains the form of the solutions at the singular points of the total differential equation system arising in the motion of an infinite homogeneous medium with velocity field $v_i = a_i(t)x_i$.

R. G. Langebartel (Urbana, Ill.)

7079:

García, Godofredo. Laws of variation of density as a function of distance for star aggregates and for planets. Actas Acad. Nac. Ci. Lima **22** (1959), no. 1/2, 17–28. (Spanish)

7080:

Das Gupta, Santi Ranjan. On the exact solution of transfer equation in the Milne-Eddington model: conservative, grey isotropic scattering. Indian J. Theoret. Phys. **6** (1958), 39–50.

7081:

Giovanelli, R. G. Radiative transfer in non-uniform media. Austral. J. Phys. **12** (1959), 164–170.

The author derives the following equation of radiative transfer for an inhomogeneous medium:

$$\frac{1}{3}\nabla^2 J = \frac{1}{3\kappa}\nabla J\cdot\nabla\kappa + \kappa^2(\lambda J\cdot 4\pi S),$$

where J the total intensity, κ the attenuation coefficient, λ the ratio of absorption and attenuation coefficient, and S is the source function. He considers a semi-infinite medium bounded by the plane $z=0$, and studies solutions of the form

$$J = \sum_{n=0}^{\infty} \mathcal{J}_n(z)\cos nlx,$$

where l is a constant.

The applications of the work to stellar atmospheres is briefly discussed, and the important part played by the inhomogeneities is emphasised. *W. B. Bonnor* (London)

7082:

Pitteway, M. L. V. Reflexion levels and coupling regions in a horizontally stratified ionosphere. Philos. Trans. Roy. Soc. London. Ser. A **252** (1959), 53–68.

Problems of propagation of radio waves through a "horizontally stratified" ionosphere are commonly dealt with by a ray theory, due to Booker [same Trans. **237** (1938), 411–451], which involves the solution of a quartic equation whose roots are somewhat analogous to the refractive index in a homogeneous medium. Whenever two of these roots are equal there is coupling between two of the characteristic waves and the ray theory breaks down.

This paper gives the results of calculations, made with the help of the EDSAC, of the electron plasma frequency and the collision frequency at which this coupling occurs. They are exhibited graphically as functions of the angle of incidence for typical values of the Earth's magnetic field and vertical planes of propagation, the magnetic dip being 60°.

{In the opinion of the reviewer the practical value of investigations such as these, where the anisotropic ionosphere is regarded as horizontally stratified, is uncertain. He has pointed out [Austral. J. Sci. Res. Ser. A. **2** (1949), 169–183; MR **11**, 760] that the refractive index structure is an intrinsic property of each ray so that it is not generally possible to regard the medium as stratified. A proper ray theory must be associated with the bicharacteristics of the full wave equations, an approach which leads to an extension of Hamiltonian optics. Although Fermat's principle results from this approach in the case of an isotropic medium, it does not follow in the present case. In this respect the recent work of Haselgrove [*Physics of the ionosphere*, pp. 355–364, Physical Society of London, 1955] must still be regarded as defective.}

K. C. Westfold (Sydney)

GEOPHYSICS

See also 6862, 6881, 6897.

7083:

Crapper, G. D. A three-dimensional solution for waves in the lee of mountains. J. Fluid Mech. **6** (1959), 51–76.

Small 3-dimensional perturbations produced by a mountain in a statically stable stratified air stream are considered. If V is the main-stream velocity, β the parameter governing the static stability of the atmosphere, and g the acceleration of gravity, then the fundamental solution for a doublet disturbance in an air stream with $g\beta/V^2$ constant is obtained. The solution is extended to the case of a circular mountain. The upper boundary condition is determined rigorously by assuming small frictional forces, or, alternatively, a time dependence. The discussion shows that the "rigid lid" upper boundary condition, with the height of the lid becoming infinite, is essentially correct.

The doublet solution is not entirely satisfactory as a representation of the effect of a mountain. A circular mountain can give rise to waves having greater amplitude than the waves produced by an infinite ridge under the same conditions. *G. C. McVittie* (Urbana, Ill.)

7084:

Liu, D. T. Wave propagation in a liquid layer. Geophysics **24** (1959), 658–666.

7085:

Collins, Francis. Plane compressional Voigt waves. Geophysics **25** (1960), 483–504.

"The Voigt wave equation, long considered a possible mathematical model for seismic waves, was extensively studied in 1943 by Norman Ricker, who developed an asymptotic solution applicable at sufficiently large distances from the shotpoint. Ricker's solution does not agree with his field results (1953) in all respects. For example, Van Melle (1954) has shown that Ricker's wavelets decay too rapidly with distance to be consistent with the experimental data. This particular difference between theory and observation may arise from the form of the wave source implicitly assumed by Ricker, a doublet impulse of displacement.

A single impulse of pressure seems a more reasonable first approximation to the shot. For this source a complete solution for plane waves is developed, valid for all distances and times. A method similar to Ricker's is then outlined for obtaining an asymptotic solution suitable for computations at large distances.

The wavelets from the pressure impulse do not decay as rapidly as Ricker's doublet displacement solutions. This is a move toward better agreement with experiment. On the other hand, the pressure impulse wavelets have only a single lobe, a definite move away from the observations. There is some reason to believe, however, that the corresponding spherical waves would be more oscillatory than the plane waves.

More mathematical work is needed for further tests of the Voigt solid as a theoretical model for earth waves. The next step suggested is the computation of spherical waves from a pressure impulse." *Author's summary*

7086:

Knopoff, Leon; and Gilbert, Freeman. First motion methods in theoretical seismology. J. Acoust. Soc. Amer. **31** (1959), 1161–1168.

"Techniques are presented for approximating integral solutions to some problems in theoretical seismology. The approximations obtained are the first terms of asymptotic series in powers of $t-t_0$, where t is the time and t_0 is an arrival time. The approximations are obtained by evaluating the integral form of the Laplace transform of the time solution by the saddle point method or a variation of it. To the resulting expression is applied a Tauberian limit theorem from which is obtained the time solution. Two examples are given which illustrate some of the specific techniques for the use of the method." (Author's summary) *I. Tolstoy* (Dobbs Ferry, N.Y.)

7087:

Hanna, Nabil O. M. The induction of electric currents in non isotropic and non uniform thin spherical shells and its application to a problem in geomagnetism. Proc. Math. Phys. Soc. Egypt. no. 22 (1958), 101–108 (1959). (Arabic summary)

7088:

Ingraham, R. Theory of the cosmic ray equator. Nuovo Cimento (10) **12** (1959), 356–368. (Italian summary)

On the assumption that the atmosphere of the earth does not rotate rigidly with the earth and that this rotation drops off with increasing altitude, the author is able to show using an exact calculation that there is a correct westward shift of the cosmic ray equator from the geometric equator. *H. Messel* (Sydney)

7089:

Piddington, J. H. The transmission of geomagnetic disturbances through the atmosphere and interplanetary space. Geophys. J. **2** (1959), 173–189.

The Maxwell equations and equations of motion are written down for a mixture of ionised plasma and neutral gas. A uniform magnetic field M_0 is assumed to be applied. The equations are linearised, and plane wave solutions are found for the velocities and the deviation of the magnetic field from M_0. The dispersion relation is given, and limiting forms derived. The results are applied to a discussion of the propagation of electromagnetic fields in a model of the earth's atmosphere extending out to about 10 radii. *A. Herzenberg* (Manchester)

7090:

Cook, A. H. Reports on the progress of geophysics developments in dynamical geodesy. Geophys. J. **2** (1959), 222–240.

7091:

Ansermet, A. Sur l'application de la théorie de l'équivalence lors du calcul d'ellipsoïdes d'erreur. Schweiz. Z. Vermessg. Kulturtech. Photogr. **58** (1960), 65–76.

7092:

Bragard, L. Méthodes de détermination du géoïde. Bull. Géodésique (N.S.) no. 50 (1958), 31–35.

A number of approaches on determining geoidal heights above a reference ellipsoid are summarized. These can be grouped into two theoretical methods—a direct calculation, and the use of a cogeoid as an intermediary surface. A further distinction is made between three basic data sources—gravity anomalies, anomalies of mean curvature of principal sections, and anomalies of gravity gradients. The author believes that the direct method is superior to the indirect because of the complications involved in computing the correction between gravity on the geoid and cogeoid. Since the direct method requires knowledge of topographic densities, he advocates the use of gravity gradient anomalies as a technique which best supplies this information. *B. Chovitz* (Washington, D.C.)

7093:

Kirmser, P. G.; and Wakabayashi, I. Triangulation—a precise method for satellite tracking. J. Franklin Inst. **268** (1959), 337–351.

Two rays from fixed ground points P_1 and P_2 to a point of a satellite orbit overdetermine its position. In addition to the possibility of using this overdetermination to reduce the error of the calculated position by an adjustment, it is possible to combine a ray from P_1 with the set of rays from P_2 which pass through a short section of the orbit. The redundancy can be used to pick out that ray from P_2 which goes to the same orbit point as P_1 (and hence to make it possible to dispense with a separate timing apparatus at P_2).

The pairing of rays from the two stations is carried out by considering the great circle on the celestial sphere which represents the plane passing through P_1 and P_2 and containing the ray from P_1. This great circle is drawn on the photograph taken from P_2; the intersection with the trace of the orbit is the desired ray from P_2. The problem is discussed in the gnomonic projection.

J. A. O'Keefe (Chevy Chase, Md.)

7094:

Andersen, Einar. Transfer of geographical coordinates by means of Clarke's curve of alignment. Geodaet. Inst. Skr. (3) **30** (1959), 53 pp.

Eine der Hauptaufgaben der Geodäsie besteht darin, daß die geographischen Positionen eines Punktes A sowie das Azimut und die Bogenlänge einer Flächenkurve des Rotationsellipsoides zu einem 2. Punkt B vorgegeben und die geographischen Positionen dieses Punktes gesucht sind. Statt des üblichen Brauches, als Flächenkurve die geodätische Linie zu benutzen, verwendet der Verfasser jene Kurve, die als "Feldlinie" oder "Absteckungskurve" bezeichnet wird: In jedem ihrer Punkte P liegt die Flächennormale in der durch APB gehenden Ebene. Der praktische Vorteil besteht darin, daß dann eine Korrektion der zwischen den Normalschnitten gemessenen Winkel nicht mehr erforderlich ist. Verf. stellt die Kurven-Gleichung auf, berechnet die Bogenlänge und nimmt die Koordinaten-Übertragung vor. Zahlenbeispiele dienen der praktischen Demonstration ($S=870$ km).

H. Wolf (Bonn)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 6315, 6672, 7127.

7095:

Bródy, András. Savings, computed on a price basis and savings computed on the basis of a technological matrix. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 11–14. (Hungarian. Russian and English summaries)

The author considers a Leontief type input-output model. He assumes that one industry can affect a saving in the use of the output of another industry. He shows that the saving for the total economy is less than the saving for the industry considered.

E. Lukacs (Washington, D.C.)

7096:

Uzawa, Hirofumi. Prices of the factors of production in international trade. Econometrica **27** (1959), 448–468.

The problem dealt with in this article is whether international trade leads to an equalization of wages and other price factors in the participating countries. It is shown that such equalization takes place only under very restrictive assumptions. *T. Haavelmo* (Oslo)

7097:

Cherubino, Salvatore. Sull'evoluzione economica. Principi di economia astratta. Rend. Mat. e Appl. (5) **17** (1958), 231–261.

This paper contains several generalisations of dynamic input-output systems (without reference to Leontief); for example, matrices whose elements are functions of time; a translation of a linear growth model (without reference to von Neumann) into linear programming terms; and a discussion of rates of production, of profit, of surplus value and of organic composition of capital. It is often difficult to see the economic wood for the mathematical trees; thus, it appears neither very convenient nor very plausible to postulate that rates of price changes are linear transformations of quantities and vice versa. The formal treatment, however, is interesting throughout.

G. Morton (London)

7098:

Ara, Kenjiro. The aggregation problem in input-output analysis. Econometrica **27** (1959), 257–262.

Two theorems are given. The first gives the economic interpretation of the Hatanaka-Malinvaud condition for the sectoral aggregation in input-output analysis; the second proves that the "acceptable" aggregation gives rise to the same largest characteristic root.

S. Ichimura (Osaka)

7099:

Fisher, Franklin M. Generalization of the rank and order conditions for identifiability. Econometrica **27** (1959), 431–447.

Let $Ax(t) = u(t)$ be a system of linear stochastic equations and let A_1, the first row of A, be subject to a set of restrictions. Let T be a row vector conformable with A. Then T is admissible if TA satisfies the restrictions on A_1. If the restrictions on A_1 are all homogeneous, the following definitions refer to rays of vectors, if not, to individual vectors. The author defines A_1 to be uniquely, finitely, or countably identifiable if the set of admissible vectors contains the appropriate number of elements. He gives various conditions which are necessary or sufficient for these types of identifiability to hold. In the linear case, these reduce to those of Koopmans, Rubin and Leipnik [*Statistical inference in dynamic economic models*, pp. 53–237, Wiley, New York, 1950; MR **12**, 431].

H. Rubin (East Lansing, Mich.)

7100:

Senju, Shizuo. A probabilistic approach to economic maintenance. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **6**, 58–62 (1959).

Assuming that the following quantities are known: (1) a function $f(t) + \varepsilon_i(t)$ of time t expressing the operating loss of machine i per unit length of time, where $f(t)$ is the mean loss of all machines at time t and $\varepsilon_i(t)$ is the deviation of loss of machine i from the mean value $f(t)$; (2) the maintenance cost a_i for machine i, which may also include the time loss during maintenance; the paper considers three cases: (1) the expected time average of the total loss and maintenance cost is minimized; (2) each machine is repaired when the individual loss reaches a common value l_0; (3) the operating loss is a linear function of time for methods (1) and (2).

W. W. Leutert (New York, N.Y.)

7101:

Marschak, Thomas. Centralization and decentralization in economic organizations. Econometrica **27** (1959), 399–430.

Arguments for and against centralized decision making are discussed. It is shown that, in order to settle this question, one must first agree on how to evaluate "advantages" and "disadvantages" connected with a given decision scheme. The definition of these concepts may be complicated. Certain criteria for ranking the goodness of decision schemes are indicated. *T. Haavelmo* (Oslo)

7102:

Beyer, Otfried. Über zwei Anwendungen des Satzes von Ljapounoff der Wahrscheinlichkeitsrechnung. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **2** (1958), 7–9.

The author shows the manner through which the central limit theorem of probability theory could be applied to intermediate control in the production-line manufacture of engines in order to find out whether measure-deviations of single pieces are happening at random or systematically. An explanation of the same theorem in technology, especially for estimating the repairing-plan of a set of engines—taking into account the different lifetime which every single piece may have—is also indicated.

O. Onicescu (Bucharest)

7103:

Harary, Frank. Graph theoretic methods in the management sciences. Management Sci. **5** (1959), 387–403.

This paper surveys work exemplifying a graph-theoretic approach to problems in the social sciences. The main concepts discussed are 'Liaison persons', 'Cliques', 'Strengthening and weakening group members', 'Redundancies', 'Status and contrastatus', 'Tournaments', 'Production schedules', 'Social power', and 'Structural

balance'. Several mathematically interesting results are stated. There are no proofs, but reference is made to the original papers by the author, and others.

A. M. Duquid (Providence, R.I.)

7104:

Thompson, S. P.; and Ziffer, A. J. The watchdog and the burglar. Naval Res. Logist. Quart. **6** (1959), 165–172.

The solution of a 3 by 3 matrix game, whose coefficients involve parameters. *H. Scarf* (New Haven, Conn.)

7105:

Black, Guy; and Proschan, Frank. On optimal redundancy. Operations Res. **7** (1959), 581–588.

The following model is considered. Several types of spare items are to be purchased in varying quantities, to replace items which fail while being used. If specific failure time distributions are assumed then it is possible to compute, for any type of item i and any initial purchase n_i, the probability $P_i(n_i)$ that in a fixed time the demand for this item will not exceed n_i. The authors are interested in maximizing $\prod P_i(n_i)$, or equivalently $\sum \log P_i(n_i)$, subject to a cost constraint. The main result of this paper is that if the failure time distributions are exponential $\sum \log P_i(n_i)$ is a concave function of $(n_1, \cdots, n_k)$, so that Lagrange multipliers (or the Kuhn-Tucker theorem) may be used. The second named author has found that a similar result is correct whenever the failure distributions have a monotone likelihood ratio in translations, i.e.,

$$f(t_1-w_1)/f(t_1-w_2) \geq f(t_2-w_1)/f(t_2-w_2),$$

for $t_1<t_2$ and $w_1<w_2$. A full development of this latter result is to be published elsewhere.

H. Scarf (New Haven, Conn.)

7106:

Kometani, Eiji; and Sasaki, Tsuna. A safety index for traffic with linear spacings. Mem. Fac. Engrg. Kyoto Univ. **21** (1959), 229–246; also published as Operations Res. **7** (1959), 704–720.

This paper appears in exactly the same form in both of the above journals and is an extension of a model proposed by the authors [J. Operations Res. Soc. Japan **2** (1958), 11–26; MR **21** #1905] and by Chandler, Herman and Montroll [Operations Res. **7** (1959), 86–106; MR **20** #770]. Here it is postulated that the velocity of some jth car at time t is a linear function of both the spacing between it and the car ahead, and also the velocity of the car ahead, both evaluated at a time $t-T$ with T some reaction time. The new feature is the dependence on the velocity of the lead car. Stability of the motion for sinusoidal variations in velocity is considered in detail.

G. Newell (Providence, R.I.)

7107:

Gazis, Denos; Herman, Robert; and Maradudin, Alexei. The problem of the amber signal light in traffic flow. Operations Res. **8** (1960), 112–132.

When a traffic light changes from green to amber, a driver approaching the intersection must decide whether he should try to stop or should proceed through the intersection. For any given car velocity there is a certain minimum distance in which a car can be safely brought to a stop and a certain maximum distance that it can travel (with or without acceleration) during any given amber period. If, when the light changes, the driver is closer to the intersection than the former distance he obviously should continue but if he is further from the intersection than the latter distance then he should stop to avoid being in the intersection after the light has turned red. If the amber light is of short duration, however, a driver may be caught in a "dilemma zone" in which he can neither stop safely nor clear the intersection before the light turns red. The sad result which the authors found from studying 70 intersections was that only one had an amber period long enough to avoid the dilemma zone and furthermore the recommended settings described in traffic engineering books were inadequate. They have not yet made a test case of their interesting conclusions in court, however.

G. Newell (Providence, R.I.)

7108:

van Klinken, J. The theory of random processes and actuarial statistics. Dependent and independent probabilities. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 139–162. (German, French and Italian summaries)

The author considers problems of the following type. Let us consider two groups of people, A and B. At time t the respective group sizes are $n_A(t)$ and $n_B(t)$. Given the intensities for transitions $\mu_{AB}(t)$, $\mu_{BA}(t)$, and for leaving the system $\mu_A(t)$ and $\mu_B(t)$, we can write down the usual difference-differential equation for the joint probability distribution of $n_A(t)$ and $n_B(t)$. Passing to the joint characteristic function we get instead a linear, first order partial differential equation.

While these equations are not generally solved by quadratures, it is shown that in some situations of interest for the actuary one can get satisfactory approximations, and especially approximations for the means and variances.

Having decided on such an approximation the question arises how to use empirical data from insurance practice in order to estimate the intensities or other relevant parameters. The author also discusses certain tests of hypotheses, e.g., when the Poisson approximation is used.

Similar ideas from the theory of stochastic processes are also used to predict the future development of numbers in various groups or of the present value of pensions to be paid by the insurer. *U. Grenander* (Stockholm)

7109:

★Segerdahl, Carl-Otto. A survey of results in the collective theory of risk. Probability and statistics: The Harald Cramér volume (edited by Ulf Grenander), pp. 276–299. Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York; 1959. 434 pp. $12.50.

Der Verfasser gibt eine Zusammenstellung der bisher gewonnenen Resultate in der kollektiven Risikotheorie und analysiert und verschärft sie in verschiedenen Richtungen. Keine Literaturangaben. *W. Saxer* (Zürich)

7110:

Zwinggi, Ernst. Sui metodi che si possono usare per il calcolo dei premi, delle riserve e degli utili nell'assicurazione sulla vita. Giorn. Ist. Ital. Attuari **21** (1958), 1–12.

Der Verfasser verwendet für die Aufstellung der versicherungstechnischen Barwerte einer gemischten Versicherung für die Versicherungsleistungen die kontinuierliche und für die Prämie die diskontinuierliche Methode.
W. Saxer (Zürich)

7111:

Daboni, Luciano. Ancora sul problema del rinnovamento. Giorn. Ist. Ital. Attuari **21** (1958), 39–56.

Der Verfasser hat in einer früheren Arbeit [derselbe Giorn. **19** (1956), 44–62; MR **19**, 470] das Erneuerungsproblem unter der Voraussetzung untersucht, dass ausscheidende Elemente nicht sofort ersetzt werden. Er rechnete mit einer konstanten Ausscheide-Intensität. In der vorliegenden Untersuchung wird eine variable Ausscheide-Intensität angenommen. *W. Saxer* (Zürich)

7112:

Tedeschi, Bruno. Sull'investimento "ottimo" dal punto di vista della matematica finanziaria. Giorn. Ist. Ital. Attuari **21** (1958), 57–71.

The problem considered is that of "optimum" investment of a given sum when a choice exists among a designated list of securities available for purchase at stipulated prices. The obvious solution is to select the security or securities offering the highest yield rate at the prices quoted. However, if maturity dates differ, or if some of the securities provide for early redemption on the basis of drawings by lot, this solution involves the implicit assumption that the cash income can be reinvested at the same yield rate. Other possible assumptions are discussed, such as the assumption that a specified portion of the cash income is spent and the remainder reinvested. In this case the optimum investment problem becomes a problem in linear programming. *T. N. E. Greville* (Kensington, Md.)

7113:

Wagner, Harvey M. An integer linear-programming model for machine scheduling. Naval Res. Logist. Quart. **6** (1959), 131–140.

The following machine scheduling problem is stated: Given n items, each to be processed on one or more of m machines, the order of processing for an item being partially or totally specified, find the sequencing of items on the machines which minimizes the total elapsed time to complete the manufacture of all items. A quantitative model of the scheduling process is set up using variables required to be non-negative and integer-valued and to satisfy certain linear constraints, in the hope that recently developed techniques for solving linear programming problems of this form [R. E. Gomory, Bull. Amer. Math. Soc. **64** (1958), 275–278; MR **21** #1230] may allow the computational solution of the scheduling problem. The numbers of variables and constraints in the formulation depends upon the nature of the processing specifications, and in the general case is "staggering". Considerable simplifications are shown in the case that each item must be processed on all machines in the same order; for three machines, the number of constraints is about $4n$.
P. Wolfe (Santa Monica, Calif.)

7114:

Eastman, Willard L. A note on the multi-commodity warehouse problem. Management Sci. **5** (1959), 327–331.

A version of the warehouse problem [Charnes and Cooper, Operational Res. Quart. **6** (1955), 131–172] is reduced to the following. Given $a_{ij} \geqq 0$, $0 < j < i < n$, find an increasing sequence of integers $1 = t_1, t_2, \cdots, t_r = n$ which maximizes $\sum a_{t_k, t_{k+1}}$. The maximum is shown to be a_{nn}, the last of n numbers determined recursively by the relations $a_{11} = 0$, $a_{ii} = \max_{j<i} [a_{jj} + a_{ij}]$. The equivalence of this to finding the shortest route through a (special) network is noted. *F. W. Sinden* (Murray Hill, N.J.)

7115:

★Arrow, Kenneth J.; Hurwicz, Leonid; and Uzawa, Hirofumi. Studies in linear and non-linear programming. With contributions by H. B. Chenery, S. M. Johnson, S. Karlin, T. Marschak, R. M. Solow. Stanford Mathematical Studies in the Social Sciences, vol. II. Stanford University Press, Stanford, Calif., 1958. vii+229 pp. $7.50.

This is an interesting and valuable report on the activities of the "Stanford" school of mathematical economics in the area of linear and non-linear programming. It continues a new trend of bypassing the regular journals in favor of preliminary publication as research project reports (to an agency such as the Office of Naval Research, Ford Foundation or RAND Corp.) followed by direct publication in book form.

The book is divided into three parts, characterized as follows in the introduction: "In Part I, Existence Theorems, we are concerned with investigating the mathematical structure of linear and non-linear programming and with developing a general theory of programming in linear topological spaces. Part II, Gradient Method, gives extensive studies on the gradient method, which is a mathematical formulation of the market mechanism in the competitive economy and which may be used to converge to the optimum solutions of programming problems. In Part III, Methods of Linear and Quadratic Programming, solutions to particular problems of linear and non-linear programming are discussed." The largest portion of the collection of papers which forms this book is concerned with various aspects of the Arrow-Hurwicz gradient method. The reporting in general is careful and objective, e.g., in chapter 9, T. Marschak makes no bones about computational difficulties with the modified gradient method for linear programming, and H. Uzawa in chapter 12 is similarly candid in regard to "An elementary method for linear programming". Perhaps the most important application is that of chapter 15, "Non-linear programming in economic development", by H. Chenery and H. Uzawa. To the general mathematician, chapter 4, "Programming in linear spaces", by L. Hurwicz, and chapter 2, "A theorem on convex polyhedral cones", by H. Uzawa, may be of most interest. The reviewer notes with approval the name "Mazur-Bourgin (separation) theorem".

In general, the bibliographical references do not indicate very well the work of others in the same area. Thus the bibliography of chapter 4 should be supplemented by references to the work of Ky Fan [*Linear inequalities and related systems*, Ann. of Math. Studies no. 38, pp. 99–156, Princeton Univ. Press, 1956; MR **19**, 432], R. J. Duffin [ibid., pp. 157–170; MR **19**, 374], and Donald Bratton

[Cowles Commission Discussion Paper, Mathematics no. 427 (1955)]. Again, a simpler and more general proof of the basic Minkowski-Farkas lemma than that of chapter 2 is available in Charnes and Cooper, Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 914–916. In connection with chapter 13, "Price speculation under certainty", by K. Arrow and S. Karlin, the reader interested in ". . . the very considerable simplification of analysis which results from taking account of the particular structure of the problem instead of treating it mechanically as a linear programming problem", may also consult the journal Management Science ab initio and Hoffman, Jacobs, Karush, Charnes and Cooper, among others, for such simplification on similar structures.

These, however, are minor points. The book is a required addition to any reference library in mathematical economics. *A. Charnes* (Evanston, Ill.)

7116:

★**Dennis, Jack B. Mathematical programming and electrical networks.** The Technology Press of The Massachusetts Institute of Technology, Cambridge, Mass.; John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. vi+186 pp. $4.50.

This monograph gives a novel and interesting approach to the problem of finding an extremum of a function under inequality constraints, that is, the problem of mathematical programming. It is well known that important cases of this problem have an equivalent Lagrangian formulation, where the problem is to find a set of variables, some of which may be constrained to be non-negative, and which satisfy a set of equality constraints. The author establishes a physical model, in terms of electrical networks, for the Lagrangian formulation of quadratic and linear programming. Specifically, every quadratic or linear program can be represented by an electrical model containing only voltage sources, current sources, ideal diodes, linear resistors, and ideal d.c. transformers, and conversely, every such electrical network is equivalent to a pair of dual quadratic or linear programs. This gives rise to an extension to quadratic and concave programming of the duality principle of linear programming, and to elegant physical interpretations of the fundamental theorems of linear and quadratic programming.

Algorithms for tracing the breakpoint curve of an electrical network are developed, and give rise to new methods for solving quadratic and linear programs. In particular, a new algorithm is given for the capacitated network flow problem, including the well known transportation problem. These algorithms are respectively related to Dantzig's simplex method, the primal-dual method of Dantzig et al., and the method of Ford and Fulkerson for the transportation problem. Degeneracy in breakpoint tracing is dealt with by methods partly due to Dantzig, Orden, and Wolfe.

In the last chapter a method for the general programming problem is developed, in which the direction of steepest descent is determined at each iteration by solving a quadratic program. Chapter 2, 'The nature of programming problems', together with Appendices A through F, provides an excellent introduction to mathematical programming. Unfortunately there are a large number of misprints, especially in the appendices. Appendix E, 'Some properties of concave functions', is based on an incorrect lemma (E–1), but the reader familiar with convexity will not have difficulty in supplying the needed support for the main results.

A. M. Duguid (Providence, R.I.)

7117:

Simonnard, Michel A.; and Hadley, G. F. Maximum number of iterations in the transportation problem. Naval Res. Logist. Quart. 6 (1959), 125–129.

In a linear programming problem of mn unknowns and $m+n-1$ constraints the maximum number of bases is $(mn)!/(m+n-1)!\ (mn+1-m-n)!$ Not all bases need exist or be feasible. For the standard transportation problem having m destinations and n origins the number of existing (but not necessarily feasible) bases is shown to be $n^{m-1}m^{n-1}$. *W. W. Leutert* (New York, N.Y.)

7118:

Sopka, J. J. An analysis of adequate inventory levels. IBM J. Res. Develop. 3 (1959), 54–57.

The author assumes the following rather awkward model for the demand per unit time of stock: a basic component consisting of a Poisson variable plus a constant term and an extra component of demand consisting again of Poisson variables plus constant terms, these extra components occurring at time points which are themselves Poisson distributed. Several approximations are obtained for quantities of interest in inventory calculations. *H. Scarf* (New Haven, Conn.)

7119:

Flood, Merrill M. Operations research and logistics. Naval Res. Logist. Quart. 5 (1958), 323–335.

This expository paper in part gives logistic problems to which various aspects of linear programming can be applied, providing an underlying structure for many of the ideas in the latter. The author shows the broad applicability of the linear programming model which is limited in the number of factors it considers. Difficulties always exist when useful information is desired. First a discussion of various fundamental properties of the linear programming problem is given, e.g., the well known duality theorem and the fact that one may start with general inequalities and transform to the non-negativity conditions. The logistic problems include the product distribution problem, which is applied to transportation by military tankers. It is remarked that storage and handling facilities provide against the uncertainties which often are not sufficiently understood to be included in the model. Even then, the model may become too large to obtain answers from it.

Other examples given are the personnel assignment problem, also used in production-scheduling, and the scheduling (or travelling salesman) problem, which is more difficult than the previous problem if integral solutions are needed.

An inventory control and a preventive maintenance model with uncertainties included are then discussed to show the usefulness of parametrization.

T. L. Saaty (Silver Spring, Md.)

7120:

Beckmann, Martin J. An inventory policy for repair parts. Naval Res. Logist. Quart. **6** (1959), 209–220.

"Assume that the demand for a repair part is Poisson, that the costs of shortage per part and of reordering are constant, that the probability distribution of delivery times for stock replacement is arbitrary but known, and that the order size is specified at the time of delivery. The inventory equation for this problem is formulated and solved; the optimal policy is found to be of the s, S type (Section 2). . . . It is found convenient to minimize average cost in determining order size and reordering point rather than discounted future cost (Section 4). . . . In the case of exponential delivery time . . . the minimal order size is that given by the Wilson square-root formula (Section 5). These formulae are approximately valid also in the case where the initial probability of failure is somewhat higher than the conditional-failure probability at later times." (From the author's summary)

K. J. Arrow (Stanford, Calif.)

7121:

★Kalisch, G. K. Generalized quota solutions of n-person games. Contributions to the theory of games, Vol. IV, pp. 163–177. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

The author extends Shapley's theory of quota solutions. Several theorems take the view that L is a given union of line segments of the quota solution type, and give necessary conditions for L to be a solution of some game. In the other direction, using $(-1, 0)$ normalization, the following is proved: If there exists $(\omega_1, \cdots, \omega_n)$ such that $v_{ij} = \omega_i + \omega_j$ for all i, j with $\omega_i \geq -1$ for all i, and $\sum \omega_i \geq 0$ (Shapley required $\sum \omega_i = 0$), then the game is solvable. This is not stated explicitly, but follows from the more refined theorems of the paper. Some of the ideas and results are generalized to "m-quota solutions", wherein $v(s) = \sum_{i \in S} \omega_i$ for each set S with m members.

J. H. Blau (Stanford, Calif.)

7122:

Peleg, Bezalel. On the set of solvable n-person games. Bull. Amer. Math. Soc. **65** (1959), 380–383.

The author announces a new proof, which is sketched, of Gillies' result [*Contributions to the theory of games*, Vol. IV, pp. 47–85, Princeton Univ. Press, 1959; MR **21** #4850] that the games with solutions constitute a full-dimensional subset in the space of all n-person games in (0, 1) normal form. Let $T \subseteq I_n$, $|T| \geq 2$. A partial quota for T is a vector $(\rho_i)_{i \in T}$ satisfying $\rho_i + \rho_j = v_{ij}$ for all $i, j \in T$ and also $\sum_{i \in T} \rho_i \geq 1$. If $T = I_n$, then v is solvable by the result mentioned in the review above. (The author attributes this to Kalisch in the Report of an Informal Conference on the Theory of n-Person Games, Princeton University, 1953. The abstract there, not necessarily prepared by the author, states something quite different, but the theorem in question may have been intended.) If $T \neq I_n$, let $p \in I_n - T$, and define $\omega_i = \rho_i$ for $i \in T$, $\omega_p = 1 - \sum_{i \in T} \rho_i$, and $\omega_i = 0$ for all other i. The following statement is then made. "It can be shown that there is . . . at most one player for whom $\omega_i < 0$." This is false. The solution which follows is therefore not well defined. However, for the main line of reasoning, the quoted statement, and the solution depending upon it, is used only for the case $|T| = 3$, and this is true, as are also the cases $|T| = 2$ and $|T|$ odd. It remains to show that some triple (i, j, k) possesses a partial quota, and this is easily seen to be equivalent to $v_{ij} + v_{ik} + v_{jk} \geq 2$, a condition which does not reduce dimension. It is interesting that the author's condition, demanding that v be large, insures that the core of the three person subgame is empty or a single point, while Gillies insists that v be small, so that every solution is in the core. A second proof, using the "extended" theory of von Neumann and Morgenstern, is sketched. The set of solvable games obtained differs from the first set only by a set of much lower dimension. Both are quite different from the Gillies set of solvable games, as the author points out. In fact, they do not even overlap.

J. H. Blau (Stanford, Calif.)

BIOLOGY AND SOCIOLOGY

See also 7134.

7123:

Wette, R. Regressions- und Kausalanalyse in der Biologie. Metrika **2** (1959), 131–137.

This is an expository article. All papers reviewed are standard, primarily in familiar English language sources, culminating in E. L. Scott, Amer. Math. Statist. **21** (1950), 284–288 [MR **11**, 733].

C. J. Maloney (Frederick, Md.)

7124:

Rashevsky, N. Suggestion for a possible approach to molecular biology. Bull. Math. Biophys. **21** (1959), 309–326.

Assuming, as is generally believed, that the production of a new living unit (here supposed a molecule) can take place only by self-reproduction of an already existing one, the author suggests a tentative outline of a physical theory in which such reproduction is pictured in terms of a transition of a certain system between two quantum-mechanical states.

To satisfy the postulate that the spontaneous formation of a living molecule is impossible, it is assumed that the corresponding transition is a forbidden one in the quantum-mechanical sense. On the other hand, if a system corresponding to a living molecule in state $S(A)$ coexists in close proximity with two other systems consisting of the material components of that molecule [state $S(\sum B_i)$], it is shown that the total energy of the composite system may be equal to the total energy of another system composed of the broken down molecule A [$S(\sum A_i)$] and two living molecules A. The pair of states

$$(1) \qquad S(A) + 2S(\textstyle\sum B_i),$$

$$(2) \qquad S(\textstyle\sum A_i) + 2S(A)$$

would then correspond to the same doubly degenerate energy level, allowing spontaneous transition from one state to the other. If (1) goes into (2), one living unit has disintegrated, and two have been synthesized in its place, an event which is generally believed to underlie all reproduction of living matter.

A difficulty suggests itself in that an arbitrarily small disturbance (an energy of interaction) will lift the degeneracy mentioned above. States (1) and (2) being now at different energy levels, transition cannot be allowed without opening the door to the possibility of spontaneous

generation. The difficulty is resolved, according to the author, by the use of the uncertainty relation $\delta E = \hbar/\tau$, where δE is energy uncertainty, $\hbar$ Planck's constant, and τ the average life span of the state. From similar considerations, involving the probability of equality of energy levels between states (1) and (2), the conclusion is deduced that only very large molecules can be expected to be capable of self-reproduction, which is again biologically plausible.

The theoretical formulation is further applied to the self-reproduction of a schematically represented DNA molecule. The results deduced from the model suggest the "systemic" character of self-reproduction, i.e., "self-reproducing molecules exhibit their property of self-reproduction only when they are basically parts of other, more complex molecular systems".

A. Rapoport (Ann Arbor, Mich.)

7125:

Bellman, Richard; and Kalaba, Robert. Some mathematical aspects of optimal predation in ecology and boviculture. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 718–720.

The authors make use of the functional-equation techniques of dynamic programming to formulate and solve certain problems dealing with the control aspects of optimal predation and optimal breeding policies associated with boviculture. *A. A. Mullin* (Urbana, Ill.)

7126:

Schwartz, Lorraine; and Wearden, Stanley. A distribution-free asymptotic method of estimating, testing, and setting confidence limits for heritability. Biometrics **15** (1959), 227–235.

"The genetic parameter heritability indicates the extent to which phenotypic variation is controlled by gene action. . . . Many different procedures have been used to estimate heritability when the observable random variables can be assumed to be normally distributed. . . . In this paper, the expectation of a rank order statistic closely related to the Mann-Whitney U statistic is suggested as a measure of heritability when the normal assumption is not appropriate and in particular when rank within a group is a meaningful measure of the trait under consideration." (From the author's introduction)

Z. W. Birnbaum (Seattle, Wash.)

7127:

★Luce, R. Duncan. Individual choice behavior: A theoretical analysis. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xii+153 pp. $5.95.

Unified by a central mathematical model, this book explores theories of topics in laboratory psychology such as preferences among ordinary rewards and among gambles, scaling and discrimination of perceptions, recognition of signals in noise, and learning. For psychology, the book is unquestionably important, rich, and provocative, but only a brief report, not entering at all into criticism of the psychological substance, is in order here.

When an individual is offered an opportunity to choose which among the elements of a set T he prefers (or perhaps which he considers to be loudest, brightest, most likely to succeed, or the like), the individual may well show indecision, or incomplete discrimination. For example, he may express reluctance to choose at all, vacillate, or be inconsistent from trial to trial. The author seeks to express incomplete discrimination by supposing that to each finite set T of elements x there is associated a probability measure P_T defining the probability that the individual will choose each element of T. The characteristic and pervading assumption of the book is this: If $R \subset S \subset T$, the probability that an individual confined to choice within T will choose within R is the product of the probability that he will choose within S and the probability that he would choose within R if he were confined to choice within S. Formally, $P_T(R) = P_S(R)P_T(S)$. (The author prefers to weaken and complicate this relation a bit in connection with those pairs of elements between which the person discriminates perfectly.) One illuminating way to put the assumption is that $P_S(R) = P_T(R|S)$ whenever the latter is defined.

Under mild side conditions, it is easy to adduce from the main assumption a positive real-valued function v defined (up to a constant multiple) on the elements x among which the individual chooses such that, for all T and $x \in T$, $P_T(x)$ is, except for normalization, the same as $v(x)$. This "scale" simplifies and guides the theory.

The greater part of the book is occupied with ideas tying the central assumption into various specific psychological topics. It is impractical to review these here, though many bold and often surprising conclusions are reached, not all of which have yet been put to experimental test.

The mathematical workmanship is in the modern professional tradition, without being unduly pedantic. Formal mathematical errors are rare, if present at all, but there are a few disturbing informal passages like this one on p. 50: "Equation (8) suggests why there may be more than one class of scales. We began with perfectly symmetric assumptions about the two continua and were led to equation (8) which, although equally symmetric in the sense that the v-scale can be expressed in two ways that together are symmetric, is not symmetric once a decision is made which way to write it. Presumably, nature would be forced to such a choice, even if there is no a priori way of deciding which." *L. J. Savage* (Chicago, Ill.)

7128:

Bryant, Steven J.; and Marica, John G. Strategies and learning models. Psychometrika **24** (1959), 253–256.

As an alternative to Estes' stimulus sampling theory the authors propose some strategy models. Actually their simple model is a special case of stimulus sampling theory —one stimulus sampled with probability one and conditioning parameter c. Their generalized model assigns weights to success or failure on preceding trials. It is shown that inclusion of past trials does not affect the mean asymptotic response probability.

P. Suppes (Stanford, Calif.)

INFORMATION AND COMMUNICATION THEORY

See also 6603, 6638, 7132, 7133.

7129:

Gilbert, E. N.; and Moore, E. F. Variable-length binary encodings. Bell System Tech. J. **38** (1959), 933–967.

This paper is a rich storehouse of results on efficient binary encodings of "alphabets" (finite collections of symbols with associated probabilities). It explores the characteristics and interrelationships of the "prefix property" (no code-word is the start of another code-word), the "alphabetical property" (alphabetical order of symbols preserved in the numerical order of their assigned code-words), the "strong alphabetical property" (the coding transforms alphabetical order of all messages, as well as of individual symbols, into numerical order), the "self-synchronizing property" (the decoder can establish the code synchronization from any sufficiently long sample of coded text), the "unique decipherability property" (given any coded message, there is only one inverse image), the "exhaustive property" (every semi-infinite binary sequence has one and only one decipherment), etc. The Huffman binary encoding is extensively studied relative to these properties. Questions concerning decoders with unbounded delays are raised but not answered. In the emphasis on variable-length and exhaustive encodings, uniform-length codes of the comma-free type [cf. the paper reviewed below] are ignored.

S. W. Golomb (Pasadena, Calif.)

7130:

Jaynes, E. T. Note on unique decipherability. Trans. I.R.E. **IT-5** (1959), 98–102.

Using unique decipherability as a vehicle, the author points out some known approximate relationships between information theory and thermodynamics, and observes, in his conclusion, "The basic mathematical identity of these two fields has had, thus far, very little influence on the development of the other". He regards the partition function as the best avenue of mutual approach.

R. Hamming (Murray Hill, N.J.)

7131:

Stern, T. E.; and Friedland, B. Application of modular sequential circuits to single error-correcting p-nary codes. Trans. I.R.E. **IT-5** (1959), 114–123.

"It is the purpose of this paper to present systematic methods for the construction of close-packed single error-correcting multi-level codes, coders and decoders. The emphasis in this paper will be laid on simple and efficient means of constructing the coders and decoders using linear modular sequential circuits."

The methods used are basically those of a shift register with feedback for encoding and a rather complex controller to decode. The paper is clearly written and makes use of Galois field theory. *R. Hamming* (Murray Hill, N.J.)

SERVOMECHANISMS AND CONTROL

7132:

Eden, Murray. A note on error detection in noisy logical computers. Information and Control **2** (1959), 310–313.

The known ways to make an arbitrary assemblage of unreliable switching elements behave extremely reliably use far more redundancy than would be suggested by a direct analogy to the coding theorem in information theory. If the assemblage is just a binary adder, and the problem is merely detecting occurrence of errors, then use of the modulo-3 check circuit investigated by Peterson [IBM J. Res. Develop. **2** (1958), 166–168] involves more reasonable redundancy. The author of the paper under review suggests a similar multivalued logic for error-detection in an arbitrary assemblage. He notes that while this "permits a certain extension of the coding theorem to single components in probabilistic automata, it has not yet been possible to construct a model sufficient for arbitrary aggregates of components or to show that such a model cannot be found."

E. F. Moore (Murray Hill, N.J.)

7133:

Chomsky, Noam; and Miller, George A. Finite state languages. Information and Control **1** (1958), 91–112.

A finite state language is a finite or infinite set of strings of symbols generated by a finite set of rules each of which specifies the state of the system in which it can be applied, the symbol which is generated, and the state of the system after the rule is applied. The authors explore a number of equivalent formulations of finite state languages, show that the set of all finite state languages over a given vocabulary is a boolean algebra, and provide a characterization theorem for finite state languages. The paper concludes with a discussion of the techniques of computing the number of sentences of a given finite state language of a given length, and of at least a given length, and of the number of strings of sentences of a given length.

E. J. Cogan (Bronxville, N.Y.)

7134:

George, F. H. Inductive machines and the problem of learning. Cybernetica **2** (1959), 109–126.

In this application of cybernetics the object is to produce a blueprint for a computer which can perform deductively and inductively like a human being and to show how an inductive system can exhibit learning. Necessary and sufficient conditions for self-adapting systems are sought. By playing against itself, a machine may be viewed in certain respects as a "thinking" human. A discussion of what instructions are needed to operate such a machine is given. One must consider problems of storage, frequency, recency and value of the information to be provided. In playing a game with such a machine, moves may be assumed to be reinforced only in terms of their occurrence in winning. A method of defining perception (which is also approximately recognition) as a relation on subsets to sets is given.

A discussion of the relation between problem solving and learning is given, followed by a set of sufficient conditions for the latter, i.e., classification, storage, counting, selective reinforcement and approximation methods.

This points to the need for a realization of the functions necessary for adaptations which take place, leading to an organism performing operationally as a human being, but within less deterministic principles than those described in the paper. This can be achieved only approximately.

T. L. Saaty (Silver Spring, Md.)

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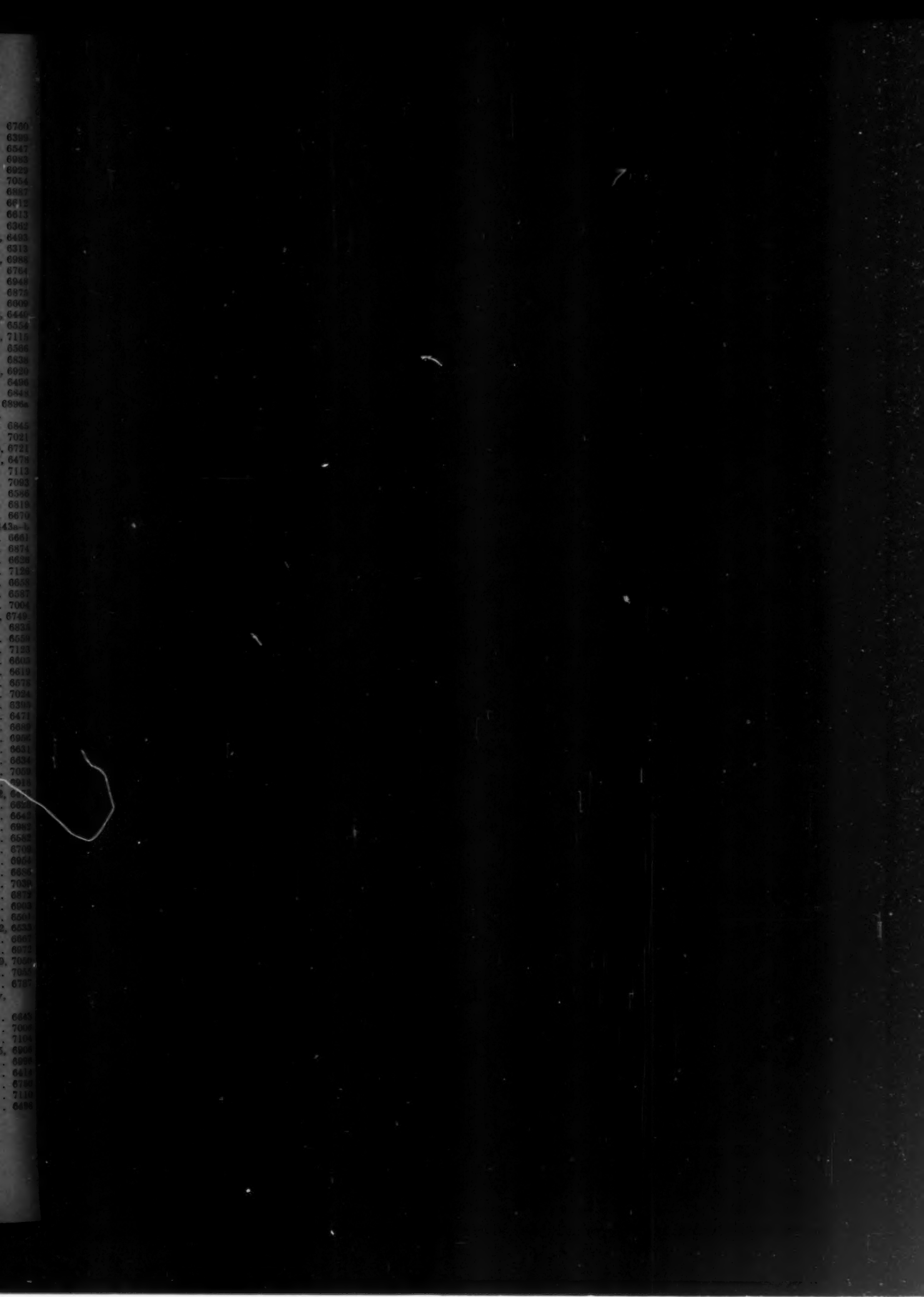
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